1 May 2005, Electromagnetism, Problem 2

1.1 (a)

The resistivity is the inverse of the conductivity, and for a cylindrical piece of material it generates a resistance $R_{inner} = \frac{L}{\sigma \pi a^2}$. Therefore, the current is $I = \frac{V}{R + \frac{L}{\sigma \pi a^2}}$. We know the magnetic field due to a constant current in a straight wire, or otherwise we can get it easily from Ampere's law. Defining the z direction along the axis of the wire, and defining the positive z direction as the direction of the current on the outer conductor, we have:

$$\boldsymbol{B} = -\frac{\mu_0 V}{2\pi\rho\left(R + \frac{L}{\sigma \pi a^2}\right)}\hat{\phi}$$

The electric field in the space between the two conductors is generated by the potential difference between the conductors. Since there are no charges in that region, the potential satisfies $\nabla^2 V = 0$. Since there is symmetry around the axis of the cable, there should not be any dependence on ϕ:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = -\frac{\partial^2 V}{\partial z^2}$$

We will guess that V can be written as $V = P(\rho)Z(z)$, so that the equation becomes:

$$\frac{1}{P\rho}\frac{\partial}{\partial \rho}(\rho P') = -\frac{Z''}{Z}$$

The boundary conditions for the solution to this equation will be:

$$V(\rho = b) = 0$$

$$V(\rho = a) = V\left(\frac{R\sigma\pi a^2 + z}{R\sigma\pi a^2 + L}\right)$$

The second one comes from defining $z = 0$ at the bottom of the conductor. Then at height z the potential is the original potential V minus the "potential loss" $IR_{inner}(z)$, and this resistance is given by the same formula that we said above except with $L - z$ instead of L. So V should be linear, and we will choose the separation constant to be 0 for that purpose. The solution for P turns out to be:

$$P = A ln\rho + B$$

It should vanish at $\rho = b$, because the potential of the outer conductor should be 0. The full solution, then, is:

$$V(\rho, z) = V\frac{ln(\rho/b)}{ln(a/b)}\frac{R\sigma\pi a^2 + z}{R\sigma\pi a^2 + L}$$

Taking minus the gradient gives the electric field:

$$\boldsymbol{E} = \frac{V}{\rho ln(a/b)} \frac{R\sigma\pi a^2 + z}{R\sigma\pi a^2 + L}\hat{\rho} - \frac{V}{R\sigma\pi a^2 + L} \frac{ln(\rho/b)}{ln(a/b)}\hat{z}$$

The electromagnetic momentum density is:

$$\frac{d\boldsymbol{p_{EB}}}{d\tau} = \epsilon_0 \boldsymbol{E} x \boldsymbol{B}$$

Then you have to integrate this over a surface of constant z to get the momentum per unit length:

$$\frac{d\boldsymbol{p_{EB}}}{dz} = \frac{\mu_0 \epsilon_0 V^2 \sigma\pi a^2 (R\sigma\pi a^2 + z)}{(R\sigma\pi a^2 + L)^2}\hat{z} \tag{1}$$

1.2 (b)

If you integrate the above expression over z running from 0 to L, you will get the total electromagnetic momentum. By momentum conservation, the mechanical momentum must be the opposite of that expression. Doing that, we get the answer:

$$\boldsymbol{p_{mech}} = -\frac{\mu_0 \epsilon_0 V^2 \sigma\pi a^2 L (R\sigma\pi a^2 + L/2)}{(R\sigma\pi a^2 + L)^2}\hat{z} \tag{2}$$

Is it correct to assume momentum conservation??