1 May 2005, Electromagnetism, Problem 2

1.1 (a)

The resistivity is the inverse of the conductivity, and for a cylindrical piece of material it generates a resistance $R_{\text{inner}} = \frac{L}{\sigma \pi a^2}$. Therefore, the current is $I = \frac{V}{R + \frac{L}{\sigma \pi a^2}}$. We know the magnetic field due to a constant current in a straight wire, or otherwise we can get it easily from Ampere’s law. Defining the $z$ direction along the axis of the wire, and defining the positive $z$ direction as the direction of the current on the outer conductor, we have:

$$B = -\frac{\mu_0 V}{2\pi \rho \left( R + \frac{L}{\sigma \pi a^2} \right)} \hat{\phi}$$

The electric field in the space between the two conductors is generated by the potential difference between the conductors. Since there are no charges in that region, the potential satisfies $\nabla^2 V = 0$. Since there is symmetry around the axis of the cable, there should not be any dependence on $\phi$:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) = -\frac{\partial^2 V}{\partial z^2}$$

We will guess that $V$ can be written as $V = P(\rho) Z(z)$, so that the equation becomes:

$$\frac{1}{P \rho} \frac{\partial}{\partial \rho} (\rho P') = -\frac{Z''}{Z}$$

The boundary conditions for the solution to this equation will be:

$$V(\rho = b) = 0$$

$$V(\rho = a) = V \left( \frac{R \sigma \pi a^2 + z}{R \sigma \pi a^2 + L} \right)$$

The second one comes from defining $z = 0$ at the bottom of the conductor. Then at height $z$ the potential is the original potential $V$ minus the “potential loss” $IR_{\text{inner}}(z)$, and this resistance is given by the same formula that we said above except with $L - z$ instead of $L$. So $V$ should be linear, and we will choose the separation constant to be 0 for that purpose. The solution for $P$ turns out to be:

$$P = A \ln \rho + B$$

It should vanish at $\rho = b$, because the potential of the outer conductor should be 0. The full solution, then, is:

$$V(\rho, z) = V \frac{\ln(\rho/b)}{\ln(a/b)} \frac{R \sigma \pi a^2 + z}{R \sigma \pi a^2 + L}$$

Taking minus the gradient gives the electric field:
\[ E = \frac{V}{\rho \ln(a/b)} \frac{R\sigma \pi a^2 + z}{R\sigma \pi a^2 + L} \hat{\rho} - \frac{V}{R\sigma \pi a^2 + L \ln(a/b)} \hat{z} \]

The electromagnetic momentum density is:

\[ \frac{d\mathbf{p}_{EB}}{d\tau} = \epsilon_0 \mathbf{E} \times \mathbf{B} \]

Then you have to integrate this over a surface of constant \( z \) to get the momentum per unit length:

\[ \frac{d\mathbf{p}_{EB}}{dz} = \mu_0 \epsilon_0 V^2 \sigma \pi a^2 (R\sigma \pi a^2 + z) \frac{(R\sigma \pi a^2 + L^2)}{(R\sigma \pi a^2 + L)^2} \hat{z} \]

(1)

1.2 (b)

If you integrate the above expression over \( z \) running from 0 to \( L \), you will get the total electromagnetic momentum. By momentum conservation, the mechanical momentum must be the opposite of that expression. Doing that, we get the answer:

\[ p_{mech} = -\frac{\mu_0 \epsilon_0 V^2 \sigma \pi a^2 L(R\sigma \pi a^2 + L/2)}{(R\sigma \pi a^2 + L)^2} \hat{z} \]

(2)

Is it correct to assume momentum conservation??