May 2004 SM #2

a. $E(\omega) = \hbar \omega n(\omega)$

$$E(\omega) = \hbar\omega \frac{1}{e^{\beta\varepsilon}-1} \qquad \varepsilon = \hbar c K = \hbar c \left(\frac{\rho}{\gamma}\omega^2\right)^{1/3}$$

$$E(\omega) = \frac{\hbar\omega}{\exp(\beta\hbar c(\rho\omega^2/\gamma)^{1/3})-1}$$

b. $$E = \frac{A}{(2\pi)^2}\int_0^\infty E(K)\, 2\pi K\, dK \qquad \omega^2 = \frac{\gamma}{\rho}K^3$$

$$\frac{E}{A} = \frac{1}{2\pi}\int_0^\infty E(\omega)\frac{2}{3}\left(\frac{\rho}{\gamma}\right)^{2/3}\omega^{1/3}\,d\omega \qquad 2\omega\, d\omega = \frac{\gamma}{\rho}3K^2 dK$$

$$\frac{2}{3}\left(\frac{\gamma}{\rho}K^3\right)^{1/2} d\omega = \frac{\gamma}{\rho}K^2 dK$$

$$\frac{2}{3}\sqrt{\frac{\rho}{\gamma}K}\; d\omega = K\,dK$$

$$\frac{2}{3}\left(\frac{\rho}{\gamma}\right)^{2/3}\omega^{1/3}\,d\omega = K\,dK$$

$$\frac{E}{A} = \frac{1}{3\pi}\int_0^\infty \frac{\hbar\omega}{\exp(\beta\hbar c(\rho\omega^2/\gamma)^{1/3})-1}\left(\frac{\rho}{\gamma}\right)^{2/3}\omega^{1/3}\,d\omega$$

Let $x = (\beta\hbar c)^{3/2}\left(\frac{\rho}{\gamma}\right)^{1/2}\omega$

$$dx = (\beta\hbar c)^{3/2}\left(\frac{\rho}{\gamma}\right)^{1/2}d\omega$$

$$x^{4/3} = (\beta\hbar c)^2\left(\frac{\rho}{\gamma}\right)^{2/3}\omega^{4/3}$$

$$\frac{E}{A} = \frac{1}{3\pi}\left(\frac{\rho}{\gamma}\right)^{2/3}\int_0^\infty \frac{1}{\exp(x^{2/3})-1}\;\frac{x^{4/3}}{(\beta\hbar c)^2(\rho/\gamma)^{2/3}}\;\frac{dx}{(\beta\hbar c)^{3/2}(\rho/\gamma)^{1/2}}$$

$$\frac{E}{A} = \frac{1}{3\pi}\left(\frac{\gamma}{\rho}\right)^{1/2}\left(\frac{K_B T}{\hbar c}\right)^{7/2}\int_0^\infty \frac{1}{\exp(x^{2/3})-1}x^{4/3}\,dx$$

$$\frac{C}{A} = \frac{\partial}{\partial T}\left(\frac{E}{A}\right) = \frac{7}{6\pi}\frac{K_B}{\hbar c}\left(\frac{K_B T}{\hbar c}\right)^{5/2}\left(\frac{\gamma}{\rho}\right)^{1/2}\int_0^\infty \frac{1}{\exp(x^{2/3})-1}x^{4/3}\,dx$$

c. As $T \to \infty$ $e^{\beta\varepsilon} \to 1 + \beta\varepsilon$

$$E(\omega) = \hbar\omega\frac{1}{1+\beta\varepsilon-1} = \hbar\omega\frac{K_B T}{\varepsilon}$$

$\therefore \quad \frac{E}{A} \sim T$

Thus the heat capacity per unit area is a constant at high temperatures.