May 2004 #2 (SM)

·Must assume wave is described in terms of harmonic **oscillator**

dispersion relation $\omega^2 = \frac{\gamma}{\rho} k^3$

a. surface in equilibrium at temperature T

wave at frequency ω: $E = (n + \frac{1}{2})\hbar\omega \quad n = 0, 1, 2, \ldots$

Partition Function for a single mode:

$$Z = \sum_n e^{-\beta E_n} = e^{-\frac{1}{2}\beta\hbar\omega} \sum_{n=0}^{\infty} e^{-\beta n \hbar\omega} = e^{-\frac{1}{2}\beta\hbar\omega} \cdot \frac{1}{1 - e^{-\beta\hbar\omega}}$$

$$\ln Z = -\tfrac{1}{2}\beta\hbar\omega - \ln\left(1 - e^{-\beta\hbar\omega}\right)$$

$$\bar{E} = -\frac{\partial}{\partial\beta}\ln Z = \tfrac{1}{2}\hbar\omega + \frac{\partial}{\partial\beta}\ln\left(1 - e^{-\beta\hbar\omega}\right)$$

Ignoring zero-point energy,

$$\bar{E} = \frac{e^{-\beta\hbar\omega}(\hbar\omega)}{1 - e^{-\beta\hbar\omega}} = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

b. Need $\sigma(\omega)d\omega$: The number of normal modes in the frequency range between ω and $\omega + d\omega$

Density of states in 2D k-space: $\sigma(\vec{k})d^2\vec{k} = \frac{A}{4\pi^2} d^2\vec{k}$ (Good for waves in a homogeneous medium)

$$\rightarrow \sigma(k)dk = \frac{A}{4\pi^2} \cdot 2\pi k\, dk = \frac{A}{2\pi} k\, dk$$

$$\sigma(\omega)d\omega = \frac{A}{2\pi} k(\omega) \left|\frac{dk}{d\omega}\right| d\omega \qquad k = \left(\frac{\rho}{\gamma}\right)^{1/3} \omega^{2/3}$$

$$\frac{dk}{d\omega} = \frac{2}{3}\left(\frac{\rho}{\gamma}\right)^{1/3} \omega^{-1/3}$$

$$\sigma(\omega)d\omega = \frac{A}{3\pi}\left(\frac{\rho}{\gamma}\right)^{2/3} \omega^{1/3}\, d\omega$$

$$\frac{\bar{E}}{A} = \int_0^{\omega_D} \frac{\hbar}{3\pi}\left(\frac{\rho}{\gamma}\right)^{2/3} \frac{\omega^{4/3}}{e^{\beta\hbar\omega} - 1}\, d\omega$$

where we have approximated the density of normal modes as having a sharp cutoff ω_D, determined by $N = \int_0^{\omega_D} \sigma(\omega)\, d\omega$, N = # of normal modes

$$C_v = \frac{\partial \bar{E}}{\partial T} = -\frac{1}{kT^2}\frac{\partial \bar{E}}{\partial \beta}$$

$$\frac{C_v}{A} = +\frac{1}{kT^2} \cdot \frac{\hbar^2}{3\pi}\left(\frac{\rho}{\gamma}\right)^{2/3}\int_0^{\omega_D} \frac{e^{\beta\hbar\omega}\,\omega^{7/3}}{(e^{\beta\hbar\omega}-1)^2}\,d\omega$$

At low temperatures, where $\beta\hbar\omega >> 1$ even for relatively low frequencies, $\omega << \omega_D$, the integrand is quite small. Thus the upper limit can be extended to ∞ without much error.

$$\beta\hbar\omega = x \qquad d\omega = \beta^{-1}\hbar^{-1} dx$$

For low T, $$\frac{\bar{E}}{A} = \frac{\hbar}{3\pi}\left(\frac{\rho}{\gamma}\right)^{2/3} \cdot \left(\frac{kT}{\hbar}\right)^{7/3}\int_0^{\infty} \frac{x^{4/3}}{e^x - 1}\,dx$$

$$\frac{C_v}{A} = \frac{k_B^{7/3}}{3\pi\hbar^{4/3}}\left(\frac{\rho}{\gamma}\right)^{2/3} T^{4/3}\int_0^{\infty}\frac{e^x\,x^{7/3}}{(e^x-1)^2}\,dx$$

c. High temperature: $\longrightarrow$ classical statistics; in particular, the equipartition theorem holds as the energy is quadratic in the displacement;

$$\frac{\bar{E}}{A} \sim \frac{N}{A}kT$$

$$\frac{C_v}{A} \sim \frac{Nk}{A} \quad \text{independent of temperature}$$