

M04T.2

(a) We will model the transverse motion of each surface wave as a 1D simple harmonic oscillator with angular frequency ω . The energy levels are $\hbar\omega n$ for $n = 0, 1, 2, 3, \dots$ where we ignore the zero point energy. We will let N denote the total number of waves. More precisely, it is the total occupation number of all the states.

The partition function for one wave is $Z = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \frac{1}{1-e^{-\beta\hbar\omega}}$, where $\beta = 1/k_B T$.

The average energy of one wave is thus $\langle E \rangle = -\frac{\partial}{\partial\beta} \log Z = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$.

(b) The density of states per area A for 2D waves with single polarization is given by

$$dN/A = \frac{1}{4\pi^2} d^2 k = \frac{k}{2\pi} dk.$$

The total energy is given by $U/A = \int_0^{k=k_F} dN/A \langle E(\omega) \rangle$, where k_F is some cut-off given by $\int_0^{k=k_F} dN = N$. Substituting the density of states and $\langle E(\omega) \rangle$ gives $U/A = \frac{\hbar}{2\pi} \int_0^{k_F} dk \frac{k\omega}{e^{\beta\hbar\omega} - 1}$.

Now, the dispersion relation $\omega^2 = \frac{\gamma}{\rho} k^3$ gives $k\omega dk = \frac{2}{3} \left(\frac{\rho}{\gamma}\right)^{2/3} \omega^{4/3} d\omega$. Substituting this into our integral gives $U/A = \frac{\hbar}{3\pi} \left(\frac{\rho}{\gamma}\right)^{2/3} \int_0^{\omega_F} d\omega \frac{\omega^{4/3}}{e^{\beta\hbar\omega} - 1}$.

Since we are interested in the low temperature limit where $\beta \rightarrow \infty$, we see that the integrand is exponentially suppressed for large ω . It is thus reasonable to extend the region of integration to $0 < \omega < \infty$. We also do a change of variables from ω to $x = \beta\hbar\omega$. This gives us

$$U/A = \frac{\hbar}{3\pi} \left(\frac{\rho}{\gamma}\right)^{2/3} \left(\frac{k_B T}{\hbar}\right)^{7/3} \int_0^{\infty} dx \frac{x^{4/3}}{e^x - 1}.$$

The heat capacity per area is given by $C/A = \frac{1}{A} \frac{\partial U}{\partial T} = \frac{7\hbar}{9\pi} \left(\frac{\rho^2 k_B^7 T^4}{\gamma^2 \hbar^7}\right)^{1/3} \int_0^{\infty} dx \frac{x^{4/3}}{e^x - 1}$.

(c) In the high temperature limit where $\beta \rightarrow 0$, we must still integrate up to a cutoff k_F . However, the integrand for the energy U simplifies:

$$U/A = \int_0^{k=k_F} dN/A \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} = \int_0^{k=k_F} dN/A \frac{\hbar\omega}{\beta\hbar\omega + O(\beta^2)} \approx \frac{1}{\beta} \int_0^{k=k_F} dN/A = kTN/A$$

follows that $C/A = kN/A$. This is essentially the equipartition theorem applied to a system with two quadratic degrees of freedom (i.e. kinetic + potential energy of a 1D SHO). In particular, C/A does not depend on T .

| Assigned groups: ssgubser_classes_F2013

One thought on "M04T.2"



December 10, 2013 at 2:41 am

I mainly agree with your solution.

Only some points about part (c). It's not clear what is the physical meaning of N . In the Debye model for solids the meaning of the cut-off was clear, -- it was the number of atoms in the crystal, and it gave the upper cut-off for the number of modes. What is the physical meaning of N in our case, i.e. which N should I take to calculate the heat capacity at high T for the actual surface of the liquid?

Another point is about relation to the equipartition theorem in (c). Although the result is definitely related, the coefficient before kN is not related to the dimensionality of the surface. Dimensionality sits inside of N . In the Debye's model the analogous answer is $3kN$ and there 3 is the number of polarizations of the wave, not the dimensionality of the crystal.
