1 May 2004, Quantum Mechanics, Problem 3

1.1 (a)

The scattering amplitude is the Fourier transform of the potential:

$$f = C \int V(\boldsymbol{x_0}) e^{-i(\boldsymbol{k_{out}} - \boldsymbol{k_{in}}) \cdot \boldsymbol{x_0}} \, d^3 x_0$$

where C is some proportionality constant and $|\boldsymbol{k_{in}}| = |\boldsymbol{k_{out}}| = k = \frac{\sqrt{2mE}}{\hbar}$. First we'll figure out the units. The differential cross section is:

$$\frac{d\sigma}{d\Omega} = |f|^2 \rightarrow L^2$$

$$\rightarrow [f] = L$$

$$[f] = [C] \frac{ML^5}{T^2} = L$$

$$[C] = \frac{T^2}{ML^4} = \frac{[m]}{[\hbar]^2}$$

$$f = -\frac{m}{2\pi\hbar^2} \int V(\boldsymbol{x_0}) e^{-i(\boldsymbol{k_{out}} - \boldsymbol{k_{in}}) \cdot \boldsymbol{x_0}} \, d^3 x_0$$

Now we must evaluate the integral:

$$\int_{-L}^{L} \int_{-L}^{L} \int_{-L}^{L} v e^{i(kz - k'_x x - k'_y y - k'_z z)} \, dx \, dy \, dz = v \frac{8 sin(k'_x L) sin(k'_y L) sin[(k'_z - k)L]}{k'_x k'_y (k'_z - k)}$$

$$f = -\frac{m}{\pi\hbar^2} v \frac{4 sin(k'_x L) sin(k'_y L) sin[(k'_z - k)L]}{k'_x k'_y (k'_z - k)}$$

$$\frac{d\sigma}{d\Omega}(\theta, \phi) = \frac{2v^2\hbar^2 sin^2(\sqrt{2mE} sin\theta cos\phi L/\hbar) sin^2(\sqrt{2mE} sin\theta sin\phi L/\hbar) sin^2[\sqrt{2mE}(cos\theta - 1)L/\hbar]}{mE^3\pi^2[sin^2\theta cos\phi sin\phi(cos\theta - 1)]^2} \tag{1}$$

You will all agree with me that this is impossible to integrate in a finite lifetime, so I will assume that when the problem said they wanted $\sigma(\theta, \phi)$, they actually meant the differential cross-section.

1.2 (b)

The Born approximation is valid when the incoming wave function is not altered significantly by the potential. In this case, we can say that the approximation is valid for low values of the potential v, or more concretely, for $v << E$.