May 2004 Preliminary Exam, Quantum Mechanics Problem 3

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Problem (Scattering from a Cube Potential):

A beam of particles of mass m and energy E propagates along the z axis of a coordinate system, and scatters from the cubic potential

$$V = \begin{cases} \text{v} & \text{if } |x| \leq L, |y| \leq L, |z| \leq L \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where v is a small constant energy.

(a) Use the Born approximation to find an explicit formula for the scattering cross section $\sigma(\theta, \phi)$ as a function of angles θ and ϕ.

(b) Under what circumstances is this approximation for the scattering cross section valid? Explain.

Solution:

(a) Using the Born approximation, we can solve for the differential cross section (same as "scattering cross section" described above because of the angular dependencies, just different terminology):

$$\sigma(\theta, \phi) = |f(\theta, \phi)|^2 \tag{2}$$

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int d^3\vec{r}\ V(\vec{r}) \exp\Big(i(\vec{k} - \vec{k}') \cdot \vec{r}\Big) \tag{3}$$

I want to integrate in Cartesian coordinates, so I will write $\vec{k} - \vec{k}'$ in these coordinates (following the hint). Without loss of generality, orient $\vec{k}$ in the $\hat{z}$ direction:

$$\begin{aligned} \implies f(\theta, \phi) &= -\frac{mv}{2\pi\hbar^2} \int_{-L}^{L} dx \int_{-L}^{L} dy \int_{-L}^{L} dz \exp\Big(i(-k'_x x - k'_y y + (k - k'_z) z)\Big) \\ &= -\frac{mv}{2\pi\hbar^2} \int_{-L}^{L} dx \exp\big(-ik'_x x\big) \int_{-L}^{L} dy \exp\Big(-ik'_y y\Big) \int_{-L}^{L} dz \exp\big(i(k - k'_z) z\big) \\ &= -\frac{mv}{2\pi\hbar^2} \left(\frac{2\sin(k'_x L)}{k'_x}\right) \left(\frac{2\sin\Big(k'_y L\Big)}{k'_y}\right) \left(\frac{2\sin((k'_z - k)L)}{(k'_z - k)}\right) \end{aligned} \tag{4}$$

Write $k'_x = k\sin(\theta)\cos(\phi)$, $k'_y = k\sin(\theta)\sin(\phi)$, and $k'_z - k = k(\cos(\theta) - 1)$:

$$f(\theta, \phi) = -\frac{4mv}{\pi\hbar^2} \left(\frac{\sin(kL\sin(\theta)\cos(\phi))}{k\sin(\theta)\cos(\phi)}\right) \left(\frac{\sin(kL\sin(\theta)\sin(\phi))}{k\sin(\theta)\sin(\phi)}\right) \left(\frac{\sin(kL(\cos(\theta) - 1))}{k(\cos(\theta) - 1)}\right) \tag{5}$$

$$\begin{aligned} \implies \sigma(\theta, \phi) &= \frac{16m^2v^2}{\pi^2\hbar^4k^6} \frac{\sin^2(kL\sin(\theta)\cos(\phi))\sin^2(kL\sin(\theta)\sin(\phi))\sin^2(kL(\cos(\theta) - 1)}{\sin^4(\theta)\sin^2(\phi)\cos^2(\phi)(\cos(\theta) - 1)^2} \\ &= \frac{2v^2\hbar^2}{\pi^2 mE^3} \frac{\sin^2(kL\sin(\theta)\cos(\phi))\sin^2(kL\sin(\theta)\sin(\phi))\sin^2(kL(\cos(\theta) - 1)}{\sin^4(\theta)\sin^2(\phi)\cos^2(\phi)(\cos(\theta) - 1)^2} \\ &= \frac{v^2\hbar^2}{2\pi^2 mE^3} \frac{\sin^2(kL\sin(\theta)\cos(\phi))\sin^2(kL\sin(\theta)\sin(\phi))\sin^2(kL(\cos(\theta) - 1)}{\sin^4(\theta)\sin^4(\frac{\theta}{2})\sin^2(\phi)\cos^2(\phi)} \end{aligned} \tag{6}$$

(b) For the Born approximation to be appropriate, we need the potential to be small in energy compared to the energy of the particle and small in size compared to wave length of the scattered wave function:

$$|v| \ll E \tag{7}$$

$$L^2 \ll \frac{1}{k^2} \sim \frac{\hbar^2}{mE} \tag{8}$$