The Hamiltonian is:

$$\mathcal{H} = a\vec{S}_1 \cdot \vec{S}_2$$

Which we can rewrite using the dot product of the spin operator:

$$\mathcal{H} = \frac{a}{2}\left[J^2 - S_1^2 - S_2^2\right]$$

Particle 1 has spin S, particle 2 has spin 1/2, so we will get:

$$E = \frac{a}{2}\left[j(j+1)\hbar^2 - S(S+1)\hbar^2 - \frac{3\hbar^2}{4}\right]$$

The total spin $j = S \pm 1/2$:

$$E = \frac{a\hbar^2}{2}\left[\left(S \pm \frac{1}{2}\right)\left(S \pm \frac{1}{2} + 1\right) - S(S+1) - \frac{3}{4}\right]$$

$$E = \frac{a\hbar^2}{2}\left[S^2 \pm \frac{S}{2} + S \pm \frac{S}{2} + \frac{1}{4} \pm \frac{1}{2} - S^2 - S - \frac{3}{4}\right] = \frac{a\hbar^2}{2}\left[\pm S \pm \frac{1}{2} - \frac{1}{2}\right]$$

The energy $S + \frac{1}{2}$ (thus taking on $2S+2$ states), while $j = S - \frac{1}{2}$, so it takes on $2S$ states.

Since we have two spin 1/2 particles, we can express every state as the up or down of each of the two particles:

$$|1,1\rangle = |{+}{+}\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle + |{-}{+}\rangle\right)$$

$$|1,-1\rangle = |{-}{-}\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle - |{-}{+}\rangle\right)$$

We apply the hamiltonian (and use our knowledge of Pauli Matrices to apply S_x and S_y to S_z eigenstates):

$$\mathcal{H}|1,1\rangle = a\left(S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z\right)|1,1\rangle$$

$$\mathcal{H}|1,1\rangle = a\left(\frac{\hbar^2}{4}|{-}{-}\rangle - \frac{\hbar^2}{4}|{-}{-}\rangle + \frac{\hbar^2}{4}|{+}{+}\rangle\right) = \frac{a\hbar^2}{4}|1,1\rangle$$

And:

$$\mathcal{H}|1,-1\rangle = a\left(\frac{\hbar^2}{4}|{+}{+}\rangle - \frac{\hbar^2}{4}|{+}{+}\rangle + \frac{\hbar^2}{4}|{-}{-}\rangle\right) = \frac{a\hbar^2}{4}|1,-1\rangle$$

Next:

$$\mathcal{H}|1,0\rangle = \frac{a}{\sqrt{2}}\left(\frac{\hbar^2}{4}|{-}{+}\rangle + \frac{\hbar^2}{4}|{+}{-}\rangle + \frac{\hbar^2}{4}|{-}{+}\rangle + \frac{\hbar^2}{4}|{+}{-}\rangle - \frac{\hbar^2}{4}|{+}{-}\rangle - \frac{\hbar^2}{4}|{-}{+}\rangle\right)$$

$$\mathcal{H}|1,0\rangle = \frac{a\hbar^2}{4}|1,0\rangle$$

Lastly:

$$\mathcal{H}\left|0,0\right\rangle = \frac{a}{\sqrt{2}}\left(\frac{\hbar^2}{4}\left|-+\right\rangle - \frac{\hbar^2}{4}\left|+-\right\rangle + \frac{\hbar^2}{4}\left|-+\right\rangle - \frac{\hbar^2}{4}\left|+-\right\rangle - \frac{\hbar^2}{4}\left|+-\right\rangle + \frac{\hbar^2}{4}\left|-+\right\rangle\right)$$

$$\mathcal{H}\left|0,0\right\rangle = -\frac{3a\hbar^2}{4}\left|0,0\right\rangle$$

So the eigenvalue $\left|1,-1\right\rangle$, while the eigenvalue $\left|0,0\right\rangle$.

We use the same eigenvectors as they are all eigenvectors ofS^z. Finding eigenvalues:

$$\mathcal{H}_B\left|1,1\right\rangle = \left(\frac{a\hbar^2}{4} + b\left(S_1^z - S_2^z\right)\right)\left|++\right\rangle = \left(\frac{a\hbar^2}{4} + b\left(\frac{\hbar}{2} - \frac{\hbar}{2}\right)\right)\left|++\right\rangle$$

$$\mathcal{H}_B\left|1,1\right\rangle = \frac{a\hbar^2}{4}\left|1,1\right\rangle$$

Similarly:

$$\mathcal{H}_B\left|1,-1\right\rangle = \left(\frac{a\hbar^2}{4} + b\left(-\frac{\hbar}{2} + \frac{\hbar}{2}\right)\right)\left|1,-1\right\rangle = \frac{a\hbar^2}{4}\left|1,-1\right\rangle$$

For $\left|1,0\right\rangle$:

$$\mathcal{H}_B\left|1,0\right\rangle = \frac{a\hbar^2}{4}\left|1,0\right\rangle + \frac{b}{\sqrt{2}}\left(\left(\frac{\hbar}{2}\left|+-\right\rangle - \frac{\hbar}{2}\left|-+\right\rangle\right) - \left(-\frac{\hbar}{2}\left|+-\right\rangle + \frac{\hbar}{2}\left|-+\right\rangle\right)\right)$$

$$\mathcal{H}_B\left|1,0\right\rangle = \frac{a\hbar^2}{4}\left|1,0\right\rangle + b\hbar\left|0,0\right\rangle$$

For $\left|0,0\right\rangle$:

$$\mathcal{H}_B\left|0,0\right\rangle = -\frac{3a\hbar^2}{4}\left|0,0\right\rangle + \frac{b}{\sqrt{2}}\left(\left(\frac{\hbar}{2}\left|+-\right\rangle + \frac{\hbar}{2}\left|-+\right\rangle\right) - \left(-\frac{\hbar}{2}\left|+-\right\rangle - \frac{\hbar}{2}\left|-+\right\rangle\right)\right)$$

$$\mathcal{H}_B\left|0,0\right\rangle = -\frac{3a\hbar^2}{4}\left|0,0\right\rangle + b\hbar\left|1,0\right\rangle$$

If we only consider these two states, then:

$$\mathcal{H}_B = \begin{pmatrix} \frac{a\hbar^2}{4} & amp; b\hbar \\ b\hbar & amp; -\frac{3a\hbar^2}{4} \end{pmatrix}$$

Which gives us the determinant:

$$\begin{vmatrix} \lambda - \frac{a\hbar^2}{4} & amp; -b\hbar \\ -b\hbar & amp; \lambda + \frac{3a\hbar^2}{4} \end{vmatrix} = 0$$

$$\left(\lambda - \frac{a\hbar^2}{4}\right)\left(\lambda + \frac{3a\hbar^2}{4}\right) - b^2\hbar^2 = 0$$

Which has solutions $\lambda_- = -a\hbar^2/4 - \hbar\sqrt{b^2 + a^2\hbar^2/4}$. I will not find the normalized eigenstates because it is not fun.