May 2004 #2 (QM)

one particle, spin $S$, other spin $\frac{1}{2}$

a. $H = a\vec{S}_1 \cdot \vec{S}_2 = \frac{a}{2}\left[J^2 - S_1^2 - S_2^2\right]$ where $\vec{J} = \vec{S}_1 + \vec{S}_2$

eigenstates: $|J, m; S, \tfrac{1}{2}\rangle$: $\quad |S+\tfrac{1}{2}, m\rangle$, $\quad |S-\tfrac{1}{2}, m\rangle$

multiplicity: $2(S+\tfrac{1}{2})+1 = 2S+2$ $\qquad$ multiplicity: $2(S-\tfrac{1}{2})+1 = 2S$

eigenvalues: $E_+ = \frac{a}{2}\hbar^2\left[(S+\tfrac{1}{2})(S+\tfrac{3}{2}) - S(S+1) - \tfrac{3}{4}\right] = \frac{a}{2}\hbar^2 \cdot S$

$E_- = \frac{a}{2}\hbar^2\left[(S-\tfrac{1}{2})(S+\tfrac{1}{2}) - S(S+1) - \tfrac{3}{4}\right] = -\frac{a}{2}\hbar^2(S+1)$

b. $S = \frac{1}{2}$:

$$\left.\begin{aligned} |1,1\rangle &= |++\rangle \\ |1,0\rangle &= \tfrac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) \\ |1,-1\rangle &= |--\rangle \end{aligned}\right\} \quad E_+ = \frac{a}{4}\hbar^2$$

$$|0,0\rangle = \tfrac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) \Big\} \quad E_- = -\frac{3}{4}a\hbar^2$$

c. $H_B = a\vec{S}_1 \cdot \vec{S}_2 + b(S_1^z - S_2^z)$

matrix of $H_B$ in total $S$ basis: in order to diagonalize & find eigenvectors

$$a\vec{S}_1 \cdot \vec{S}_2 = \begin{array}{cccc} |++\rangle & \frac{|+-\rangle + |-+\rangle}{\sqrt{2}} & |--\rangle & \frac{|+-\rangle - |-+\rangle}{\sqrt{2}} \end{array}$$
$$\begin{bmatrix} \frac{a}{4}\hbar^2 & 0 & 0 & 0 \\ 0 & \frac{a}{4}\hbar^2 & 0 & 0 \\ 0 & 0 & \frac{a}{4}\hbar^2 & 0 \\ 0 & 0 & 0 & -\frac{3}{4}a\hbar^2 \end{bmatrix}$$

$(S_1^z - S_2^z)|++\rangle = 0$

$(S_1^z - S_2^z)\left(|+-\rangle + |-+\rangle\right) = \frac{\hbar}{2}\left[|+-\rangle - |-+\rangle + |+-\rangle - |-+\rangle\right] = \hbar\left[|+-\rangle - |-+\rangle\right]$

$(S_1^z - S_2^z)|--\rangle = 0$

$(S_1^z - S_2^z)\left(|+-\rangle - |-+\rangle\right) = \frac{\hbar}{2}\left[|+-\rangle + |-+\rangle - (-|+-\rangle - |-+\rangle)\right] = \hbar\left[|+-\rangle + |-+\rangle\right)$

$$b(S_1^z - S_2^z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b\hbar \\ 0 & 0 & 0 & 0 \\ 0 & b\hbar & 0 & 0 \end{bmatrix}$$

$$H_B = \begin{bmatrix} \frac{a}{4}\hbar^2 & 0 & 0 & 0 \\ 0 & \frac{a}{4}\hbar^2 & 0 & b\hbar \\ 0 & 0 & \frac{a}{4}\hbar^2 & 0 \\ 0 & b\hbar & 0 & -\frac{3}{4}a\hbar^2 \end{bmatrix}$$

let $\frac{a}{4}\hbar^2 \equiv \alpha \qquad b\hbar \equiv \beta$

Diagonalize $H_B$:
$$\begin{vmatrix} \alpha-\lambda & 0 & 0 & 0 \\ 0 & \alpha-\lambda & 0 & \beta \\ 0 & 0 & \alpha-\lambda & 0 \\ 0 & \beta & 0 & -3\alpha-\lambda \end{vmatrix} = 0$$

$$(\alpha-\lambda)\left[(\alpha-\lambda)(\alpha-\lambda)(-3\alpha-\lambda) + \beta(-\beta)(\alpha-\lambda)\right] = 0$$

$$(\alpha-\lambda)^2\left[(\alpha-\lambda)(-3\alpha-\lambda) - \beta^2\right] = 0$$

$\lambda = \alpha, \alpha \quad$ or $\quad (\alpha-\lambda)(-3\alpha-\lambda) = \beta^2$

$(\lambda-\alpha)(\lambda+3\alpha) = \beta^2 \qquad \lambda^2 + 2\alpha\lambda - 3\alpha^2 - \beta^2 = 0$

$$\lambda = \frac{-2\alpha \pm \sqrt{4\alpha^2 + 12\alpha^2 + 4\beta^2}}{2}$$

$$\lambda = -\alpha \pm \sqrt{4\alpha^2 + \beta^2}$$

multiplicites: 2 states from the triplet are **the** same energy;
other 2 states are shifted in energy by the B-field (up from $\alpha$, down from $-3\alpha$)

For $E = \alpha = \frac{a}{4}\hbar^2$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & -4\alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \Rightarrow \begin{matrix} x_4 = 0 \\ x_2 = 0 \end{matrix} \quad \text{eigenvectors} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow |++\rangle, |--\rangle$ as before

For $\lambda_\pm$,
$$\begin{pmatrix} 2\alpha\mp\sqrt{\ } & 0 & 0 & 0 \\ 0 & 2\alpha\mp\sqrt{\ } & 0 & \beta \\ 0 & 0 & 2\alpha\mp\sqrt{\ } & 0 \\ 0 & \beta & 0 & -2\alpha\mp\sqrt{\ } \end{pmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_3 = 0 \end{matrix}$$

For $E = -\alpha + \sqrt{4\alpha^2+\beta^2}$ (triplet state)
$\beta x_2 - (2\alpha + \sqrt{4\alpha^2+\beta^2})x_4 = 0$
$$\hookrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\beta}{2\alpha+\sqrt{4\alpha^2+\beta^2}} \end{pmatrix} \text{ then normalize}$$

For $E = -\alpha - \sqrt{4\alpha^2+\beta^2}$,
$(2\alpha + \sqrt{4\alpha^2+\beta^2})x_2 + \beta x_4 = 0 \rightarrow \begin{pmatrix} 0 \\ -\frac{\beta}{2\alpha+\sqrt{4\alpha^2+\beta^2}} \\ 0 \\ 1 \end{pmatrix}$ then normalize