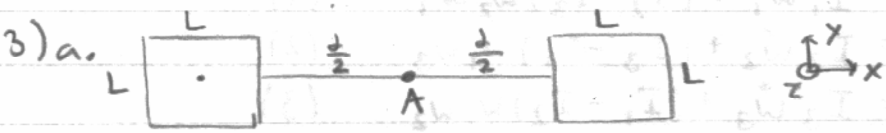
Seth Dorfman Prelims January 4, 2006

May 2004 CM

3) a.



For a single plate about its center

$$I_z = \int_{-L/2}^{L/2}\int_{L/2}^{L/2} \rho_m (x^2+y^2)\,dx\,dy$$
$$= 4\int_0^{L/2}\int_0^{L/2} \rho_m (x^2+y^2)\,dx\,dy$$
$$= 4\rho_m \int_0^{L/2} \frac{x^3}{3} + xy^2\Big|_{x=0}^{x=L/2} dy$$
$$= 4\rho_m \int_0^{L/2} \frac{(L/2)^3}{3} + \left(\frac{L}{2}\right)y^2\,dy$$
$$= 4\rho_m \left(y\frac{(L/2)^3}{3} + \left(\frac{L}{2}\right)\frac{y^3}{3} \Big|_{x=0}^{x=L/2} \right)$$
$$= 4\rho_m \left(\frac{2}{3}\left(\frac{L}{2}\right)^4 \right)$$
$$= 4\frac{m}{L^2}\cdot\frac{1}{3}\frac{L^4}{8}$$
$$I_z = \frac{1}{6}mL^2$$
$$I_x = I_y = \int_{-L/2}^{L/2}\int_{L/2}^{L/2} \rho_m x^2\,dx\,dy$$
$$= 4\rho_m \int_0^{L/2}\int_0^{L/2} x^2\,dx\,dy$$
$$= 4\rho_m \left(\frac{L}{2}\right)\frac{(L/2)^3}{3}$$
$$= 4\frac{m}{L^2}\frac{1}{3}\frac{L^4}{16}$$
$$I_x = I_y = \frac{1}{12}mL^2$$

By the parallel axis theorm, about point A:

$$I_x = \frac{1}{12}mL^2$$
$$I_y = \frac{1}{12}mL^2 + \frac{1}{4}md^2$$
$$I_z = \frac{1}{6}mL^2 + \frac{1}{4}md^2$$

Since the plates are indentical and equidistant from A we have for both:

$$I_1 = \frac{1}{3}mL^2 + \frac{1}{2}md^2$$
$$I_2 = \frac{1}{6}mL^2 + \frac{1}{2}md^2$$
$$I_3 = \frac{1}{6}mL^2$$

b. $\vec{L} = I_1 \omega_1 \hat{z} + I_2 \omega_2 \hat{y} + I_3 \omega_3 \hat{x}$

$$\frac{d\vec{L}}{dt}\Big|_{lab} = \frac{d\vec{L}}{dt}\Big|_{rot} + \vec{\omega}\times\vec{L}$$
$$\vec{\tau} = I_1\dot{\omega}_1\hat{z} + I_2\dot{\omega}_2\hat{y} + I_3\dot{\omega}_3\hat{x}$$
$$+ I_2\omega_2\omega_3\hat{z} - I_1\omega_1\omega_3\hat{y} - I_3\omega_2\omega_3\hat{z} + I_1\omega_1\omega_2\hat{x}$$
$$+ I_3\omega_1\omega_3\hat{y} - I_2\omega_1\omega_2\hat{x}$$

Or in component form:

$$\tau_z = I_1 \dot{w}_1 + (I_2 - I_3) w_2 w_3 \qquad (1)$$
$$\tau_y = I_2 \dot{w}_2 + (I_3 - I_1) w_1 w_3 \qquad (2)$$
$$\tau_x = I_3 \dot{w}_3 + (I_1 - I_2) w_1 w_2 \qquad (3)$$

$\vec{\tau} = 0$, initially only w_2 is nonzero.

Immediately after the asteroid hits, w_1 and w_3 are initially small but nonzero. Let: $w_1 = w_{10} e^{st}$ (s complex)

$$w_3 = w_{30} e^{st}$$

In (2) $w_1 w_3 \sim \varepsilon^2$ and can be neglected

thus $I_2 \dot{w}_2 \approx 0$ and w_2 can be taken as constant

$\therefore$ (1) $\Rightarrow s I_1 w_1 + (I_2 - I_3) w_2 w_3 = 0$

(3) $\Rightarrow s I_3 w_3 + (I_1 - I_2) w_1 w_2 = 0$

$$w_3 = \frac{1}{s I_3} (I_2 - I_1) w_2 w_1$$

$$s I_1 \cancel{w_1} + (I_2 - I_3) w_2 \cdot \frac{1}{s I_3} (I_2 - I_1) w_2 \cancel{w_1} = 0$$

$$s^2 I_1 I_3 + w_2^2 (I_2 - I_3)(I_2 - I_1) = 0$$

$$s^2 = -w_2^2 \frac{(I_2 - I_3)(I_2 - I_1)}{I_1 I_3}$$

Since $I_1 > I_2 > I_3$, $s^2 > 0$

Thus w_1 and w_3 represent a superposition of exponentially growing and exponentially decaying solutions.

The growing solution dominates and the motion is perturbed strongly.

$$\tau = \frac{1}{s} = w_2 \sqrt{\frac{I_1 I_3}{(I_2 - I_3)(I_1 - I_2)}}$$

Solving for w_2:

In the rotating frame:

$\vec{F} = -G \frac{m^2}{(d+L)^2}$ (Force on the center of one panel is due to only the other panel by symmetry, also approx $L \ll d$)

$$\vec{a}|_{lab} = \vec{a}_{rot} + \vec{w}_2 \times \vec{v}$$

$$-\frac{g}{6} = -G \frac{m}{(d+L)^2} - w_2 \cdot w_2 \cdot \frac{1}{2}(d + L)$$

$$\frac{g}{3(d+L)} - 2G \frac{m}{(d+L)^3} = w_2^2$$

$$w_2 = \sqrt{\frac{g}{3(d+L)} - 2G \frac{m}{(d+L)^3}}$$