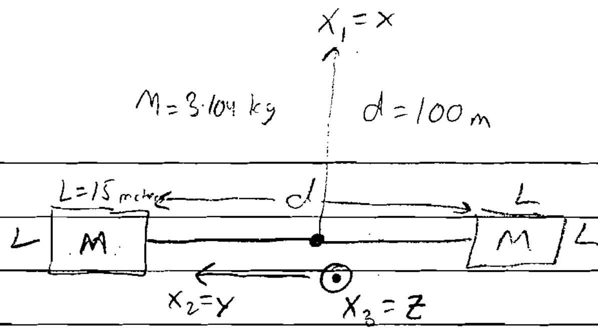
May 2004 #1 (CM)



a.

$$I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} x_k x_k - x_i x_j\right)$$

$$= \int \rho(\vec{r})\, dV \left(\delta_{ij} x_k x_k - x_i x_j\right)$$

$$\rho\, dV \to \sigma\, dx\, dy \qquad \sigma = \begin{cases} \frac{M}{L^2} & \frac{d}{2} < |y| < \frac{d}{2} + L, \quad -\frac{L}{2} < x < \frac{L}{2}, \quad z = 0 \\ 0 & \text{else} \end{cases}$$

$$I_{11} = \int \sigma\, dx\, dy \left(y^2 + z^2\right) \qquad z = 0 \qquad I_{11} = 2\sigma \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \int_{\frac{d}{2}}^{\frac{d}{2}+L} dy\, y^2$$

$$I_{11} = \frac{2\sigma L}{3} y^3 \Big|_{\frac{d}{2}}^{\frac{d}{2}+L} = \frac{2M}{3L} \cdot \left[\left(\frac{d}{2} + L\right)^3 - \left(\frac{d}{2}\right)^3\right]$$

$$\left(\frac{d}{2} + L\right)^3 = \left(\frac{d}{2}\right)^3 + 3\left(\frac{d}{2}\right)^2 L + 3\left(\frac{d}{2}\right) L^2 + L^3$$

$$\Rightarrow I_{11} = \frac{2M}{3L}\left(\frac{3}{4} d^2 L + \frac{3}{2} d L^2 + L^3\right)$$

$$I_{11} = \frac{2}{3} M L^2 + M d L + \frac{1}{2} M d^2$$

$$I_{12} = \int \sigma\, dx\, dy\, (-xy) \qquad \text{mass is symmetric about } x_1 \quad = 0$$

$$I_{13} = \int \sigma\, dx\, dy (-xz) \qquad z = 0 \text{ for all mass!} \quad I_{13} = 0$$

$+ \; I_{23} = 0$

$$I_{22} = \int \sigma\, dx\, dy \left(x^2 + \cancel{z^2}\right) = 2\sigma \int_{d/2}^{d/2+L} dy \int_{-L/2}^{L/2} dx\, x^2 = \frac{2M}{3L} \cdot x^3 \Big|_{-L/2}^{L/2}$$

$$I_{22} = \frac{4M}{3L}\left(\frac{L}{2}\right)^3 = \frac{ML^2}{6}$$

$$I_{33} = \int \sigma\, dx\, dy \left(x^2 + y^2\right) = I_{11} + I_{22}$$

$$\Rightarrow I_1 = \frac{5}{6} M L^2 + M d L + \frac{1}{2} M d^2$$

$$I_2 = \frac{2}{3} M L^2 + M d L + \frac{1}{2} M d^2$$

$$I_3 = \frac{1}{6} M L^2$$

b. pseudogravity at center of square section = $\frac{g}{6}$

$$a = \omega_2^2 r \quad a = \frac{g}{6} \quad \omega_2^2 = \frac{g}{6(\frac{d}{2}+\frac{L}{2})} = \frac{g}{3(d+L)}$$

Euler Equations

$$\left(\frac{d\vec{L}}{dt}\right)_{fixed} = \vec{N} = \vec{0} = \left(\frac{d\vec{L}}{dt}\right)_{rot} + \vec{\omega}\times\vec{L}$$

$$\dot{\vec{L}} + \vec{\omega}\times\vec{L} = 0$$

$$L_1 = I_1\omega_1 \qquad \dot{L}_1 = I_1\dot{\omega}_1 \qquad (\vec{\omega}\times\vec{L})_1 = \omega_2 L_3 - \omega_3 L_2$$
$$= I_3\omega_2\omega_3 - I_2\omega_2\omega_3$$

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0$$
$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 = 0$$
$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0$$

small perturbation to $\vec{\omega}$: $\quad \vec{\omega} = \omega_2\hat{e}_2 + \mu\hat{e}_1 + \lambda\hat{e}_3$

μ, λ small, $\quad \dot{\omega}_1 = \dot{\mu} \quad \dot{\omega}_3 = \dot{\lambda}$

Then $I_2\dot{\omega}_2 \approx 0 \Rightarrow \omega_2 \approx$ constant

$$I_1\dot{\mu} = (I_2 - I_3)\omega_2\lambda \qquad \dot{\mu} = \left(\frac{I_2 - I_3}{I_1}\omega_2\right)\lambda$$
$$I_3\dot{\lambda} = (I_1 - I_2)\omega_2\mu \qquad \dot{\lambda} = \left(\frac{I_1 - I_2}{I_3}\omega_2\right)\mu$$

$$\Rightarrow \ddot{\mu} = \frac{(I_2 - I_3)(I_1 - I_2)\omega_2^2}{I_1 I_3}\mu$$

$$\ddot{\mu} = \frac{1}{\tau^2}\mu \qquad \underline{\tau > 0 : \text{ unstable}}$$

char. timescale: $$\tau = \frac{1}{\omega_2}\sqrt{\frac{I_1 I_3}{(I_1 - I_2)(I_2 - I_3)}}$$

$$\omega_2 = \sqrt{\frac{g}{3(d+L)}}$$