1 May 2004, Mechanics, Problem 3

1.1 (a)

Let x be the axis of the system, and let y be the axis perpendicular to the axis of the object but in the plane of the figure. Then z will be the axis perpendicular to the picture. A cool result is:

$$I_x = \int y^2 dm$$

$$I_y = \int x^2 dm$$

$$I_z = \int (x^2 + y^2) dm = I_x + I_y$$

We find:

$$I_1 = I_z = m\left(d^2 + dL + \frac{5L^2}{6}\right) \tag{1}$$

$$I_2 = I_y = m\left(d^2 + dL + \frac{2L^2}{3}\right) \tag{2}$$

$$I_3 = I_x = m\frac{L^2}{6} \tag{3}$$

1.2 (b)

Just to make sure I said the axis clearly, the y-axis, which is also the 2-axis, is the axis the rotations about which look like the motion of rowing a boat. The pseudo-gravity gives us the value of the initial angular frequency:

$$\frac{d+L}{2}\bar{\omega}^2 = \frac{g}{6} \rightarrow \bar{\omega} = \sqrt{\frac{g}{3(d+L)}}$$

We need to either know Euler's equations or how to derive them. We know that the angular momentum in the rotating frame is:

$$\boldsymbol{L} = \tilde{I}\boldsymbol{\omega}$$

where the tilde means we are using the inertia tensor. Then the equation of motion will be that the applied torque plus the pseudo-torque due to the rotation is equal to the angular momentum in the moving frame:

$$\boldsymbol{\tau} - \omega \times \boldsymbol{L} = \dot{\boldsymbol{L}}$$

With zero torque, and plugging in $\boldsymbol{\omega} =< \alpha, \omega, \beta >$, we get:

$$\omega\beta(I_2 - I_3) = I_1\dot{\alpha}$$

$$-\alpha\beta I_2 = \alpha\beta(I_3 - I_1) = I_2\dot{\omega}$$

$$\alpha\omega I_3 = \alpha\omega(I_1 - I_2) = I_3\dot{\beta}$$

We are assuming that α and β are small quantities, for they are perturbations. In that case, the quantity on the left side of the second equation is second order in these small quantities, and we can take it to be 0. This implies that $\dot{\omega}$ is constant, and equal to $\bar{\omega}$. Then we have to solve the other two equations for either β or α. It turns out that:

$$\omega^2\beta(I_2 - I_3) = I_1\ddot{\beta} \tag{4}$$

Since $I_2 > I_3$, the motion will be exponential and it will diverge from the equilibrium position. The characteristic time will be:

$$\mathcal{T} = \sqrt{\frac{I_1}{\omega^2(I_2 - I_3)}} = \sqrt{\frac{(d^2 + Ld + 5L^2/6)3(d + L)}{(d^2 + dL + L^2/2)g}} \approx 6s \tag{5}$$