1 May 2004, Mechanics, Problem 1

1.1 (a)

Let's first use conservation of momentum. If the big ball were infinitely massive, the small ball would hit it with speed u and recoil with speed u. So to first order in the small quantities, we have:

$$mu - Mv = -mu + Mv_f$$
$$v_f = 2\frac{m}{M}u - v = -v\left(1 - 2\frac{mu}{Mv}\right)$$
$$\delta v = -2\frac{m}{M}u$$

The negative sign (in v, which is defined to the left) means the change in velocity is to the right, i.e., the particle is decelerating. Let's now say M=1 and understand that our new m is really $m/M << 1$. This is just for simplicity of notation. Now we can use conservation of energy to find the change in u:

$$mu^2 + v^2 = mu_f^2 + (2mu - v)^2 = mu_f^2 + v^2 - 4muv$$
$$u_f = -\sqrt{u^2 + 4uv} = -u\sqrt{1 + 4v/u} \approx -u(1 + 2v/u)$$
$$\delta u = 2v$$

The negative sign in front of the square root was chosen because, assuming the big ball is impenetrable, the small ball must go to the left after the collision. Then the positive sign in δu means that the change in velocity is to the left, i.e., the particle is accelerating. These changes in velocities take place once every period of the motion of u. That time is:

$$\delta t = \frac{2y}{u}$$

where y is the position of the big ball. We can therefore get two approximate equations:

$$\dot{v} = -mu^2/y$$
$$\dot{u} = vu/y = -\dot{y}u/y$$
$$u = u_0 d/y$$
$$-v\frac{dv}{dy} = \dot{v} = -mu_0^2 d^2/y^3$$
$$v^2 = -\frac{mu_0^2 d^2}{y^2} + v_0^2 + mu_0^2$$
$$v = \sqrt{-\frac{mu_0^2 d^2}{y^2} + v_0^2 + mu_0^2}$$

The turning point will be at v=0:

$$y = d\sqrt{\frac{mu_0^2}{Mv_0^2 + mu_0^2}} \quad (1)$$

Finally, solve for y from the equation for v:

$$-\frac{dy}{dt} = v = \sqrt{-\frac{mu_0^2 d^2}{y^2} + v_0^2 + mu_0^2}$$

$$t = -\int_d^y \frac{dy}{\sqrt{-\frac{mu_0^2 d^2}{y^2} + v_0^2 + mu_0^2}} = -\int_d^y \frac{ydy}{\sqrt{-mu_0^2 d^2 + y^2(v_0^2 + mu_0^2)}} =$$

$$= -\int_{d^2 v_0^2}^{-mu_0^2 d^2 + y^2(v_0^2+mu_0^2)} \frac{1}{2(v_0^2 + mu_0^2)} \frac{d\eta}{\sqrt{\eta}} = -\frac{1}{2(v_0^2 + mu_0^2)} \left[2\eta^{1/2}\right]_{d^2 v_0^2}^{-mu_0^2 d^2 + y^2(v_0^2+mu_0^2)} =$$

$$= -\frac{1}{(v_0^2 + mu_0^2)} \left[\sqrt{-mu_0^2 d^2 + y^2(v_0^2 + mu_0^2)} - \sqrt{d^2 v_0^2}\right]$$

$$y = \sqrt{\frac{\left[dv_0 - t(v_0^2 + mu_0^2)\right]^2 + mu_0^2 d^2}{v_0^2 + mu_0^2}}$$

$$y = \sqrt{d^2 + t^2(v_0^2 + mu_0^2) - 2dv_0 t} = \sqrt{(d - v_0 t)^2 + mu_0^2 t^2 / M} \tag{2}$$