1 May 2004, Electromagnetism, Problem 3

1.1 (a)

We will use Faraday's law:

$$\int \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt}$$

and the fact that the initial energy is $mv_0^2/2$. We will assume that we don't need any information about the fringe fields, by assuming that the radius b of the penny is small: $b << D$, so that we only need to know the field along the axis of the solenoid. So we need to find this field. Notice that having a solenoid of field B_0 is equivalent to having a magnetized bar of the same radius, with magnetization B_0/μ_0, because both have a surface current of that value. We now want to solve for the auxiliary field $\boldsymbol{H}$ everywhere along the axis of the magnetized bar. Notice that:

$$\nabla \cdot \boldsymbol{H} = \frac{1}{\mu_0}\nabla \cdot \boldsymbol{B} - \nabla \cdot \boldsymbol{M} = -\nabla \cdot \boldsymbol{M}$$

$$\boldsymbol{M} = \frac{B_0}{\mu_0}\Theta(-z)\Theta(z+L)\Theta(-\rho + D/2)\hat{z}$$

where L is the length of the solenoid. Therefore,

$$\nabla \cdot \boldsymbol{H} = -[-\frac{B_0}{\mu_0}\delta(z) + \frac{B_0}{\mu_0}\delta(z+L)]\Theta(-\rho + D/2) = \frac{B_0}{\mu_0}(\delta(z) - \delta(z+L))\Theta(D/2-\rho)$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J_f} = \boldsymbol{0}$$

Now let's look at a totally different problem. Suppose we want the electric field due to a uniform disc of charge, of radius D/2, at a distance z from the center of the disc along its axis. Then the equations we have to solve are:

$$\nabla \cdot \boldsymbol{E} = \frac{\sigma}{\epsilon_0}\delta(z)\Theta(D/2-\rho)$$

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}$$

We can find the solution by writing down the potential at the test point due to each area element of the disc and integrate over the disc. This is quite simple, and the result is:

$$\boldsymbol{E} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + (D/2)^2}}\right)\hat{z}$$

Going back to our problem with the magnetized bar, we see that our equations are exactly the same as the equations of the electric field problem, except we have two sources instead of one. By superposition, we can write down solutions for each, replacing σ/ϵ_0 by B_0/μ_0. The result is:

$$\boldsymbol{H} = \frac{B_0}{2\mu_0}\left(1 - \frac{z}{\sqrt{z^2 + (D/2)^2}}\right)\hat{z} - \frac{B_0}{2\mu_0}\left(1 - \frac{z+L}{\sqrt{(z+L)^2 + (D/2)^2}}\right)\hat{z}$$

$$= \frac{B_0}{2\mu_0}\left(-\frac{z}{\sqrt{z^2+(D/2)^2}} + \frac{z+L}{\sqrt{(z+L)^2+(D/2)^2}}\right)\hat{z}$$

And this is also the H-field for the solenoid problem, so we can easily get the B-field by multiplying by μ_0. This solution for the B-field is known as the "Pufu duality". Now we are left with the rest of the problem. By Faraday's law, at a radius r from the center of the penny, the induced electric field is given by:

$$E2\pi r = -\pi r^2 \frac{\partial B}{\partial t}$$

$$\boldsymbol{E} = -\frac{r}{2}\frac{\partial B}{\partial t}\hat{\phi}$$

and the induced volume current is:

$$\boldsymbol{J} = \frac{\boldsymbol{E}}{\rho_R}$$

The total power radiated by the penny is:

$$P = \int_0^b \int_0^\epsilon \int_0^{2\pi} \boldsymbol{J}\cdot\boldsymbol{E} r\, d\phi\, dz\, dr = \frac{\pi\epsilon}{2\rho_R}\int_0^b r^3 \left(\frac{\partial B}{\partial t}\right)^2 dr$$

where ϵ is the thickness of the penny. Assuming $b << D$, we can take the derivative of the B-field to by constant along the surface of the coin, and pull it out of the integral:

$$P = \frac{\pi\epsilon b^4}{8\rho_R}\left(\frac{\partial B}{\partial t}\right)^2 = \frac{d}{dt}\left(\frac{mv^2}{2}\right) = mv\frac{dv}{dt} = mv^2\frac{dv}{dz}$$

$$\frac{dv}{dz} = \frac{b^2}{8\rho\rho_R}\left(\frac{\partial B}{\partial z}\right)^2$$

$$-v_0 = \int_{v_0}^0 dv = \int_\infty^0 \frac{b^2}{8\rho\rho_R}\left(\frac{\partial B}{\partial z}\right)^2 dz$$

$$v_0 = \int_0^\infty \frac{b^2}{8\rho\rho_R}\left(\frac{\partial B}{\partial z}\right)^2 dz$$

$$\frac{\partial B}{\partial z} = \frac{B_0(D/2)^2}{2}\left(-\frac{1}{(z^2+(D/2)^2)^{3/2}} + \frac{1}{[(z+L)^2+(D/2)^2]^{3/2}}\right)$$

$$v_0 = \int_0^\infty \frac{b^2 B_0^2 D^4}{512\rho\rho_R}\left(-\frac{1}{(z^2+(D/2)^2)^{3/2}} + \frac{1}{[(z+L)^2+(D/2)^2]^{3/2}}\right)^2 dz$$

Now if we assume that the solenoid is infinitely long in the -z direction, then the second term inside the square drops out and we get:

$$v_0 = \int_0^\infty \frac{b^2 B_0^2 D^4}{512\rho\rho_R}\frac{1}{(z^2+(D/2)^2)^3}\, dz$$

By using a trigonometric substitution $z = (D/2)tan(\theta/2)$ we can compute this integral, and get the final answer:

$$v_0 = \frac{3\pi b^2 B_0^2}{256\rho\rho_R D} \approx 4.60\,10^{-6} m/s \tag{1}$$

where I have estimated $b \approx 5mm$.