Let us define Φ_1 to be the potential in the dielectric, and $\Phi_2$to be the potential outside the dielectric. We know that the expansion for potential in spherical coordinates is given by:

$$\Phi_1 = \sum_l \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$
$$\Phi_2 = \sum_l \left(C_l r^l + D_l r^{-(l+1)}\right) P_l(\cos\theta)$$

In order for Φ_2 to provide to $r \longrightarrow \infty$. Therefore, we must have $C_1 = -E_0$, and all other $C_l = 0$.

Because at r=a there is a chargeless conducting sphere, we must have:

$$\frac{1}{a}\left.\frac{\partial\Phi_1}{\partial\theta}\right|_{r=a} = 0$$

And so (the charge free condition keeps this true for l=0):

$$A_l = -B_l a^{-(2l+1)}$$

We lastly have the continuity equations for the dielectric sphere boundary. Continuity of $E_{||}$:

$$-\frac{1}{b}\left.\frac{\partial\Phi_1}{\partial\theta}\right|_{r=b} = -\frac{1}{b}\left.\frac{\partial\Phi_2}{\partial\theta}\right|_{r=b}$$
$$A_l b^l + B_l b^{-(l+1)} = C_l b^l + D_l b^{-(l+1)}$$

And continuity of $D_\perp$:

$$-\epsilon\left.\frac{\partial\Phi_1}{\partial r}\right|_{r=b} = -\epsilon_0\left.\frac{\partial\Phi_2}{\partial r}\right|_{r=b}$$
$$\epsilon\left(lA_l b^{l-1} - (l+1)B_l b^{-(l+2)}\right) = \epsilon_0\left(lC_l b^{l-1} - (l+1)D_l b^{-(l+2)}\right)$$

Plugging previous conditions into the $E_{||}$ condition:

$$-B_1\frac{b}{a^3} + B_1\frac{1}{b^2} = -E_0 b + D_1\frac{1}{b^2}$$
$$-B_l a^{-(2l+1)} b^l + B_l b^{-(l+1)} = D_l b^{-(l+1)}; \quad l \neq 1$$

Solving for D_l:

$$D_1 = E_0 b^3 - B_1\left(\frac{b^3}{a^3} - 1\right)$$
$$D_l = -B_l\left(\left(\frac{b}{a}\right)^{2l+1} - 1\right); \quad l \neq 1$$

Plugging all of these into the condition for $D_\perp$, first for l=1:

$$\epsilon\left(-B_1\frac{1}{a^3} - B_1\frac{2}{b^3}\right) = \epsilon_0\left(-E_0 - 2\left(E_0 b^3 - B_1\left(\frac{b^3}{a^3} - 1\right)\right)\frac{1}{b^3}\right)$$
$$\frac{\epsilon}{\epsilon_0}\left(-B_1\frac{1}{a^3} - B_1\frac{2}{b^3}\right) = -3E_0 + 2B_1\left(\frac{1}{a^3} - \frac{1}{b^3}\right)$$

Define $n = \sqrt{\epsilon/\epsilon_0}$:

$$B_1\left(\frac{n^2+2}{a^3}+2\frac{n^2-1}{b^3}\right)=3E_0$$
$$B_1=\frac{3E_0b^3a^3}{b^3\left(n^2+2\right)+2a^3\left(n^2-1\right)}$$

Define $\beta=b^3\left(n^2+2\right)+2a^3\left(n^2-1\right)$. Plugging in for the others:

$$A_1=-\frac{3E_0b^3a^3}{\beta}\frac{1}{a^3}=-\frac{3E_0b^3}{\beta}$$
$$D_1=E_0b^3-\frac{3E_0b^3a^3}{\beta}\left(\frac{b^3}{a^3}-1\right)=E_0b^3\left(1-\frac{3\left(b^3-a^3\right)}{\beta}\right)$$

Now solving $l\neq 1$ case:

$$\epsilon\left(-lB_la^{-(2l+l)}b^{l-1}-(l+1)B_lb^{-(l+2)}\right)=-\epsilon_0\left((l+1)\left(-B_l\left(\left(\frac{b}{a}\right)^{2l+1}-1\right)\right)b^{-(l+2)}\right)$$

It can be seen that these sides will never be equal unless $B_l=0$, which means:

$$A_l=-0a^{-(2l+1)}=0$$
$$D_l=-0\left(\left(\frac{b}{a}\right)^{2l+1}-1\right)=0$$

So that, in the dielectric:

$$\Phi=\frac{3E_0b^3}{\beta}\left(a^3r^{-2}-r\right)\cos\theta$$

Giving the electric field:

$$\vec{E}=\hat{r}\frac{3E_0b^3}{\beta}\left(2\left(\frac{a}{r}\right)^3+1\right)\cos\theta+\hat{\theta}\frac{3E_0b^3}{\beta}\left(\left(\frac{a}{r}\right)^3-1\right)\sin\theta$$

where $\beta=b^3\left(n^2+2\right)+2a^3\left(n^2-1\right)$.