1 May 2004, Electromagnetism, Problem 1

1.1 (a)

$$\nabla \cdot \boldsymbol{D} = \rho_f = 0$$
$$\nabla \times \boldsymbol{D} = \epsilon \nabla \times \boldsymbol{E} = \boldsymbol{0}$$

These two equations hold both in the insulator and in the space outside of it. We can therefore write:

$$\boldsymbol{D} = -\nabla \phi$$
$$\nabla^2 \phi = 0$$

where ϕ is a scalar potential. We will solve this equation in both regions and then use the boundary conditions:

$$(i) Q_{sphere} = 0$$
$$(ii) \Delta E^{||}_{sphere/ins} = 0$$
$$(iii) \Delta E^{perp}_{sphere/ins} = \frac{\sigma}{\epsilon_0}$$
$$(iv) \Delta E^{||}_{ins/out} = 0$$
$$(v) \Delta D^{perp}_{ins/out} = \sigma_f = 0$$
$$(vi) \lim_{r \to \infty} \boldsymbol{E} = E_0 \hat{z}$$

With the last equation I define the direction $\hat{z}$. The general solution to Laplace's equation looks like:

$$\phi = \sum_l (A_l r^l + B_l r^{-l-1}) P_l(cos\theta)$$

where P_l are the Legendre polynomials. We have, then, two sets of constants, because:

$$\phi_{out} = \sum_l (A_l r^l + B_l r^{-l-1}) P_l(cos\theta)$$
$$\phi_{ins} = \sum_l (\alpha_l r^l + \beta_l r^{-l-1}) P_l(cos\theta)$$

Let's start with the condition (vi). Since the field far from the sphere is $E_0 \hat{z}$, so the displacement field must also be $E_0 \hat{z}$ (we are working with $\epsilon_0 = 1$ throughout the whole problem). Therefore, the potential must be $-E_0 z = -E_0 r cos\theta$. Meanwhile:

$$\lim_{r \to \infty} \phi_{out} = \sum_l A_l r^l P_l(cos\theta) = -E_0 r cos\theta$$

We know that $P_0(cos\theta) \sim 1$ and $P_1(cos\theta) \sim cos\theta$, so this means that all modes with $l > 1$ must be 0, and by continuity of the parallel field at the boundary with the insulator, all the modes with $l > 1$ must also be 0 in the insulator region. Furthermore, this condition tells us that:

$$A_1 P_1(cos\theta) = -E_0 cos\theta$$

It looks like we could also rule out the $l = 0$ mode, but we should not do that yet, because when $r \to \infty$, the $l = 0$ mode becomes negligible with respect to the $l = 1$ mode, therefore it might be that we don't see it at $r \to \infty$, not that it doesn't exist (you'll hate me, because in reality it does turn out to be 0). So now we have the potential, with arbitrary constants ready to be solved for:

$$\phi_{out} = B_0 r^{-1} + (-E_0 r + B_1 r^{-2}) cos\theta$$
$$\phi_{ins} = \beta_0 r^{-1} + (\alpha_1 r + \beta_1 r^{-2}) cos\theta$$

Notice that I threw out A_0 and α_0, since we can always add a constant term to the potential without changing the physical results. Let's compute the displacement field:

$$\boldsymbol{D}_{out} = -\nabla\phi_{out} = [B_0 r^{-2} + (E_0 + 2B_1 r^{-3}) cos\theta]\hat{r} + (-E_0 + B_1 r^{-3}) sin\theta\hat{\theta}$$
$$\boldsymbol{D}_{ins} = -\nabla\phi_{ins} = [\beta_0 r^{-2} + (-\alpha_1 + 2\beta_1 r^{-3}) cos\theta]\hat{r} + (\alpha_1 + \beta_1 r^{-3}) sin\theta\hat{\theta}$$

Write down the boundary conditions:

$$(ii)\, \alpha_1 = -\beta_1 a^{-3}$$
$$(v)\, B_0 = \beta_0 \ and \ E_0 + 2B_1 b^{-3} = -\alpha_1 + 2\beta_1 b^{-3}$$
$$(iv)\ -E_0 + B_1 b^{-3} = \frac{\alpha_1 + \beta_1 b^{-3}}{\epsilon_r}$$
$$(iii)\, \sigma \equiv \frac{\beta_0 a^{-2} + (-\alpha_1 + 2\beta_1 a^{-3}) cos\theta}{\epsilon_r}$$
$$(i)\, Q = \int_0^{\pi}\int_0^{2\pi} \sigma a^2 sin\theta\, d\phi\, d\theta = \frac{2\pi a^2}{\epsilon_r} 2\beta_0 a^{-2} = 0 \to \beta_0 = 0 \to B_0 = 0$$

So now we have three linear equations ((ii),(v) and (iv)) in three unknowns (β_1, α_1 and B_1). This is straightforward to solve, obtaining:

$$\alpha_1 = \frac{3E_0}{-\left(\frac{2}{\epsilon_r} + 1\right) + 2\left(\frac{a}{b}\right)^3 \left(\frac{1}{\epsilon_r} - 1\right)} \tag{1}$$

$$\beta_1 = \frac{3E_0}{a^{-3}\left(\frac{2}{\epsilon_r} + 1\right) - 2b^{-3}\left(\frac{1}{\epsilon_r} - 1\right)} \tag{2}$$

$$\boldsymbol{E}_{ins} = \frac{\boldsymbol{D}_{ins}}{\epsilon_r} = \frac{(-\alpha_1 + 2\beta_1 r^{-3}) cos\theta}{\epsilon_r}\hat{r} + \frac{(\alpha_1 + \beta_1 r^{-3}) sin\theta}{\epsilon_r}\hat{\theta} \tag{3}$$