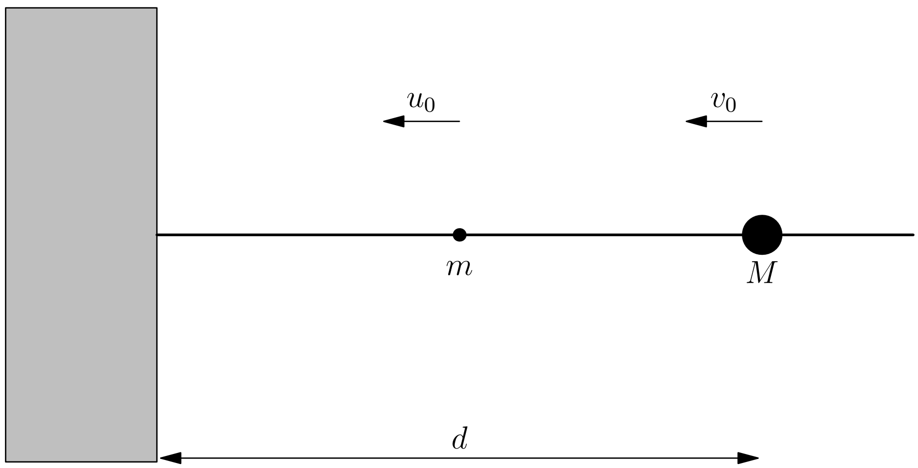
M04M.1—Particles on a Line

Problem

Two elastic spherical particles with masses m and M ($m \ll M$) are constrained to move along a straight line with an elastically reflecting wall at its end. At $t = 0$ they are in motion as shown, with $u_0 \gg v_0$.



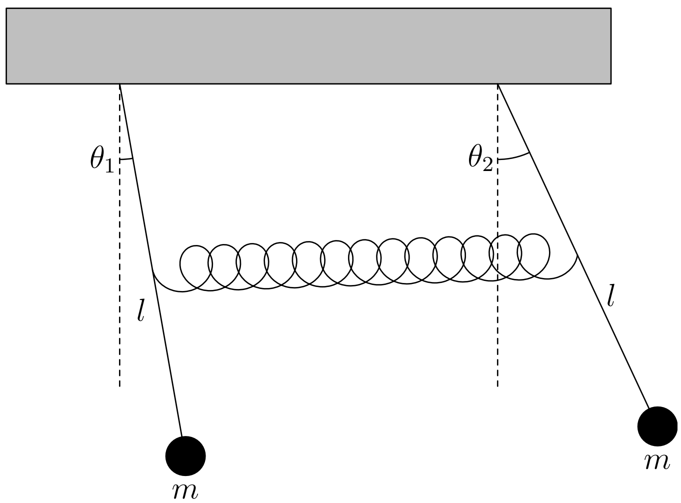
Find the subsequent motion of M, averaged over the period of motion of m. In this approximation, how far does M travel before turning around?

(Note: The initial location of m is irrelevant in this limit.)

M04M.2—Coupled Pendula

Problem

Two simple pendula, each of length l and mass m are coupled by a spring of force constant k which is attached to their massless and inextensible rods at their halfway points. The spring is relaxed when the pendula are vertical.



a) Write a Lagrangian for the system.

b) Find the normal modes and their frequencies for small oscillations about equilibrium.

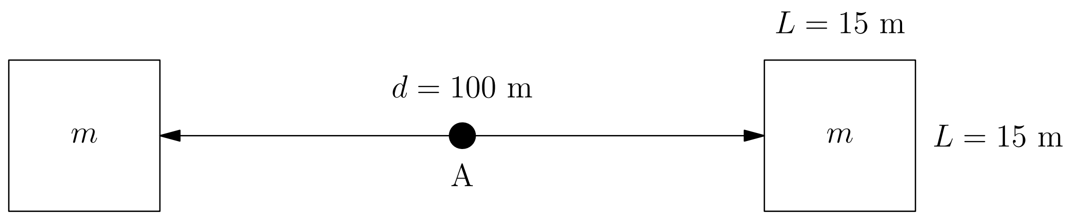
c) At $t = 0$ the left pendulum is displaced by a small angle $\theta_1(0) = \theta_0$ and released from rest while the right pendulum is at rest with $\theta_2(0) = 0$. Find $\theta_{1,2}(t)$.

d) How long will it be before the energy of the left pendulum is transferred completely to the right pendulum?

M04M.3—Panels in Space

Problem

A set of two rigidly linked panels is placed in outer space, where it can spin about 3 different axes. The panels are in the form of two thin, square, planar sections, $L = 15$ m on a side, each of mass $m = 3 \times 10^4$ kg. The two square sections are at a distance of $d = 100$ m and are connected with low-mass rigid beams (not shown in the figure).



a) Compute the principal moments of inertia $I_1 > I_2 > I_3$ about the point A.

b) After its construction, the set of panels was set spinning about the axis with the moment of inertia I_2, with its angular velocity chosen so that the pseudo-gravity at the center of each square section is $g/6$. Sadly, a tiny asteroid came by soon after and its impact nudged the angular velocity a little bit away from the "2^{nd} axis". Show that the resulting motion of the panels will be perturbed strongly. What is the characteristic time for the perturbation to grow strong?

M04E.1—Conducting Sphere Coated with Dielectric

Problem

An uncharged conducting sphere of radius a is coated with a thick insulating shell of dielectric constant ϵ_r, out to radius b. The object is placed in an initially uniform electric field E_0. What is the field everywhere in the insulator?

M04E.2—Classical Radiation from a Hydrogen Atom

Problem

In a naïve classical model of the hydrogen atom's ground state, the electron moves in a circular orbit of radius $r_0 = 0.53 \times 10^{-10}$ m around the center of mass of the electron-proton pair. Since the electron is accelerating, classically it will continuously radiate energy. Calculate the time it will take the electron to spiral into the proton, assuming its orbit is always nearly circular, and that the motion continues until the radius of the electron's orbit is reduced to that of the proton, 10^{-15} m. Are relativistic effects important for this estimate?

M04E.3—Penny Thrown into a Solenoid

Problem

A penny is thrown towards a large solenoid magnet. The penny moves along the axis of the solenoid with a frictionless constraint which keeps the plane of the penny perpendicular to the solenoid axis. As the penny approaches the solenoid, eddy currents are induced in it and result in a repulsive force which slows its motion. Estimate the minimal initial velocity which is needed in order for the penny to reach the entrance of the long solenoid magnet, with internal field of $B = 1$ T and diameter $D = 0.1$ m. You may ignore gravity and take the mass density and resistivity of copper to be $\rho \approx 10^4$ kg/m^3 and $\rho_R \approx 10^{-6}$ Ohm m, respectively.

M04Q.1—Hydrogen Molecule

Problem

Consider a hydrogen molecule H_2. Write down the Hamiltonian, keeping only the kinetic energy terms and the Coulomb interactions of all the constituents and omitting terms which cause fine and hyperfine structure.

a) What is the degeneracy of the ground state? Give all quantum numbers and symmetries of the ground state(s), including the electron and proton degrees of freedom.

b) What is the degeneracy, and what are al the quantum numbers of the first excited state of this H_2 molecule? Explain.

c) What is the energy difference between ground and first excited states? Estimate it first through a formula, in terms of properties of the molecule's ground state, and then in electron-Volts (eV).

M04Q.2—Positronium in a Magnetic Field

Problem

Consider ths spin degrees of freedom of a two-particle system, one with spin S and the other with spin 1/2. The Hamiltonian is

$$H = a\boldsymbol{S_1} \cdot \boldsymbol{S_2},$$

with a a constant. Here $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ stand for the vector spin operators of particle 1 and 2, respectively.

a) Calculate the eigenvalues of H. What are their multiplicities?

b) Consider now the special case, corresponding to the spin degrees of freedom of positronium, where both spins are S =1/2. What are the eigenvalues and corresponding eigenstates of H?

c) When the positronium is placed in a magnetic field, oriented in the z-direction, the Hamiltonian becomes

$$H_B = a\boldsymbol{S_1} \cdot \boldsymbol{S_2} + b(S_1^z - S_2^z),$$

where b is a constant. Describe the multiplicities of the resulting spectrum, and calculate the eigenvalues and eigenvectors of H_B.

M04Q.3—Scattering from a Cube Potential

Problem

A beam of particles of mass m and energy E propagates along the z axis of a coordinate system, and scatters from the cubic potential

$$V = \begin{cases} v & \text{if } |x| \leq L, |y| \leq L, |z| \leq L, \\ 0 & \text{otherwise} \end{cases}$$

where v is a small constant energy

a) Use the Born approximation to find an explicit formula for the scattering cross section $\sigma = \sigma(\theta, \phi)$ as a function of the angles θ and ϕ.
Recall that spherical coordinates of a point in space (r, θ, ϕ) are related to cartesian coordinates (x, y, z) by $x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi$ and $z = r \cos\theta$. The Born approximation is easy to evaluate in one coordinate system and hard in the other.

b) Under what circumstances is this approximation for the scattering cross section valid? Explain.

M04T.1—Equations of State

Problem

Experimentally, the equation of state (internal energy, U, and pressure, p) of a substance was found to be:

$$U = Vu(T), \quad p = \alpha\, u(T),$$

where V is the volume, T is the temperature, α is a constant and $u(T)$ is an unknown function.

a) Determine the relation of T and V in an adiabatic process. If the process is instead isothermal, how much heat is exchanged with the heat bath?

b) Sketch the p-V diagram for a Carnot cycle (2 adiabats, 2 isotherms) using this substance. Express the efficiency in terms of the function $u(T)$.

c) Explain why the above efficiency should equal $1-T_c/T_h$, where T_c and T_h are the temperatures of the low and high temperature isotherms in the Carnot cycle.

d) What conclusions can you draw about the function $u(T)$, corresponding to the given value of α?

e) Give a physical system that shows this behavior.

M04T.2—Surface Waves and Heat Capacity (M05T.3)

Problem

Consider waves on a liquid surface where the restoring force is produced by the surface tension. Assume there is a single polarization and the dispersion relation is

$$\omega^2 = \frac{\gamma}{\rho} k^3,$$

where γ is the surface tension of the liquid, ρ is its density, ω is the frequency of the waves and k is their wavenumber. Our goal is to find the contribution of these waves to the heat capacity of the liquid.

a) If the surface is in equilibrium at temperature T, what is the average energy of a wave with frequency ω? (Ignore the $\hbar\omega/2$ zero point energy.)

b) At low temperatures what are the energy per unit area and heat capacity per unit area of these surface waves? You may leave dimensionless integrals in your answer.

c) What can you say about the high temperature heat capacity per unit area?

M04T.3—Binary Mixture of Atoms

Problem

Consider a binary mixture of atoms labeled A and B. There are $N \gg 1$ atoms of which xN are of type A and $(1-x)N$ are of type B. The atoms occupy equidistant sites along a line. ignoring the kinetic eneergy, the statistical mechanics is governed by the potential energy which is determined as follows: two neighboring A atoms or two neighboring B atoms contribute $-\epsilon$ while a pair of neighboring ABs contribute $-\epsilon/2$. A sample arrangement may look like

$$\cdots \mathrm{AAABABBBAABAAA} \cdots$$

a) What is the average size of a cluster of A atoms at temperature $T = \infty$ in the limit $N \to \infty$?

b) What is the average size of a cluster of A atoms at $T = 0$ in the limit $N \to \infty$?

c) Calculate an estimate for the free energy of the mixture regarding the atoms to be independently and randomly distributed.

d) Does the above value provide a variational bound on the free energy? (Upper, or lower bound?)

e) Within this approximation, estimate the phase transition temperature at $x = 1/2$.

f) Actually, this system has no phase transition at $T = 0$. Explain why.