M03T.2 White Dwarf Star

Model a white dwarf star as a degenerate Fermi gas of electrons, supported against gravitational collapse by the electron degeneracy pressure.

(a)

The fermi wavevector k_F can be found in terms of the total number of electrons, N, as follows:

$$N = 2\,\frac{V}{(2\pi)^3} * \frac{4}{3}\pi k_F^3 \tag{1}$$

This defines the fermi energy

$$\epsilon_F = \frac{\hbar^2 k^2}{2m_e} = \frac{\hbar^2\left(\frac{N}{V}\,\frac{3}{\pi}\right)}{2m_e}\,. \tag{2}$$

The total energy is given by the integral

$$U = \int_0^{k_F} 2\,\frac{V}{(2\pi)^3}\,4\pi k^2\,\frac{\hbar^2 k^2}{2m_e}\,dk. \tag{3}$$

Thus, the total kinetic energy U_k is

$$U_k = N\,\frac{3}{5}\,\epsilon_F = \frac{3N(\hbar\pi)^2}{10m_e}\left(\frac{3M}{\pi N}\right) \tag{4}$$

(b)

The gravitational binding energy of a uniform density sphere is

$$U_{grav} = -\frac{3GM^2}{5R}. \tag{5}$$

The equilibrium radius of a white dwarf can be found by solving the equation

$$\frac{\partial}{\partial R}\left(U_k + U_{grav}\right) = 0. \tag{6}$$

Using the fact that the mass of the white dwarf $M = 2Nm_p$, we obtain the solution for R_{eq}

$$R = (\hbar\pi)^2\left(\frac{9}{4\pi^2}\right)^{2/3}\frac{1}{Gm_e m_p^{5/3}}\frac{1}{M^{1/3}} \propto M^{-1/3}. \tag{7}$$

(c)

In this part, we consider the electrons to be highly relativistic. Now the dispersion relation is given by $\epsilon = c\hbar k$.

The energy is now given by

$$U = 2\frac{V}{(2\pi)^3}\int_0^{k_F} 4\pi k^2 c\hbar k dk \tag{8}$$

$$U_k = \frac{3(\pi\hbar c)N}{4}\left(\frac{3N}{\pi V}\right)^{1/3}. \tag{9}$$

(d)

In order to find the critical mass below which a white dwarf star is stable against collapse, we solve the equation

$$\frac{\partial}{\partial R}\left(U_k + U_{grav}\right) = 0. \tag{10}$$

Using $M = 2Nm_p$, we obtain the critical mass

$$M_c^{2/3} = \frac{5\hbar\pi}{4G}\left(\frac{9}{4\pi^2}\right)^{1/3}\left(\frac{1}{2m_p}\right)^{4/3}. \tag{11}$$

One thought on “M03T.2 White Dwarf Star”

December 30, 2013 at 3:09 am

This solution is correct. Two typos: lost V in (4) and c in (11).
Also in (d) you don't really have to differentiate over R because both potential energies behave as $\frac{1}{R}$. It's enough to just look at the coefficient of this $1/R$.