M03-1 Thermodynamics of an Elastic String

$f = \mu x - \alpha T + \beta T x$

$C_x = A(x) T$

a) well $C_x = T \left(\frac{\partial S}{\partial T}\right)_x$

$\left(\frac{\partial S}{\partial T}\right)_x = A(x)$

first off $dU = T dS + f dx$

$S = T A(x)$

$dF = -S dT + f dx$

$\left(\frac{\partial F}{\partial T}\right)_x = -S$

$\left(\frac{\partial S}{\partial x}\right)\Big|_T = -\frac{\partial^2 F}{\partial T \partial x}$

$\left(\frac{\partial F}{\partial x}\right)_T = f$

$\frac{\partial f}{\partial T}\Big|_x = \frac{\partial^2 F}{\partial x \partial T} = -\frac{\partial S}{\partial x}\Big|_T$ Maxwell relation

$\left(\frac{\partial S}{\partial x}\right)_T = -\left(\frac{\partial f}{\partial T}\right)\Big|_x = -(-\alpha + \beta x)$

$\left(\frac{\partial S}{\partial x}\right)\Big|_T = \boxed{\alpha - \beta x}$

b) well $C_x = \cancel{T}\left(\frac{\partial S}{\partial T}\right)_x = A(x)\cancel{T}$

function of x only

$S = A(x) T + M(x)$

$\frac{\partial S}{\partial x}\Big|_T = T \frac{\partial A(x)}{\partial x} + \frac{\partial M(x)}{\partial x}$

$$T\frac{\partial A}{\partial x} + \frac{\partial M}{\partial x} = \left(\frac{\partial S}{\partial x}\right)_T = \alpha - \beta x$$

so since this does not depend on T, $\boxed{\frac{\partial A}{\partial x} = 0}$

c) well, $S = AT + M(x)$

$$\frac{\partial M}{\partial x} = \alpha - \beta x$$

$$M = \alpha x - \frac{1}{2}\beta x^2 + B$$

$$\boxed{S = AT + \alpha x - \frac{1}{2}\beta x^2 + B} \qquad \boxed{\left.\frac{\partial S}{\partial T}\right|_x = A}$$

$$dU = T\,dS + f\,dx$$

$$\left(\frac{\partial U}{\partial x}\right)_T = T\left(\frac{\partial S}{\partial x}\right)_T + f$$

$$\left(\frac{\partial U}{\partial x}\right)_T = \cancel{T(\alpha - \beta x)} + \mu x \cancel{- \alpha T} \cancel{+ \beta T x}$$

$$\left(\frac{\partial U}{\partial x}\right)_T = \mu x$$

$$S = AT + \alpha x - \frac{1}{2}\beta x^2 + B$$

$$dQ = T\,dS = T\left[\left(\frac{\partial S}{\partial T}\right)_f dT + \left(\frac{\partial S}{\partial f}\right)_T df\right]$$

$$dQ = \left(C_F\,dT + T\left(\frac{\partial S}{\partial f}\right)_T df\right)$$

$$dQ = C_F\,dT + T\left(\frac{\partial S}{\partial f}\right)_T\left(\left(\frac{\partial f}{\partial x}\right)_T dx + \left(\frac{\partial f}{\partial T}\right)_x dT\right)$$

$$C_x = C_F + T\left(\frac{\partial S}{\partial f}\right)_T\left(\frac{\partial f}{\partial T}\right)_x$$

$$dG = -S\,dT - x\,df$$

$$\left(\frac{\partial G}{\partial T}\right)_f = -S$$

$$\frac{\partial S}{\partial f} = -\frac{\partial^2 G}{\partial T \partial f} = -\frac{\partial}{\partial T}(-x)$$

$$\left(\frac{\partial S}{\partial f}\right)_T = \left(\frac{\partial x}{\partial T}\right)_f$$

$$f = (\mu + \beta T)\,x - \alpha T$$

$$x = \frac{f + \alpha T}{\mu + \beta T}$$

$$\left(\frac{\partial f}{\partial T}\right)_x = -\alpha + \beta x$$

$$\left(\frac{\partial S}{\partial f}\right)_T = \left(\frac{\partial x}{\partial T}\right)_f = \frac{\partial}{\partial T}\left(\frac{f+\alpha T}{\mu+\beta T}\right) = \frac{(\mu+\beta T)\alpha - (f+\alpha T)(\beta)}{(\mu+\beta T)^2}$$

$$\left(\frac{\partial x}{\partial T}\right)_f = \frac{\alpha}{\mu+\beta T} - \frac{\beta(f+\alpha T)}{(\mu+\beta T)^2}$$

so

d) $\cancel{B}\ f = \mu x - \alpha T + \beta T x$

$$x(\mu + BT) = f + \alpha T$$

$$x = \frac{f+\alpha T}{\mu+\beta T}$$

$$\cancel{As}\ S = AT + \alpha\left(\frac{f+\alpha T}{\mu+\beta T}\right) + \frac{1}{2}\beta\left(\frac{f+\alpha T}{\mu+\beta T}\right)^2 + \cancel{B}$$

$$\frac{\partial S}{\partial T}\bigg|_f = A + \alpha\,\frac{\alpha(\mu+\beta T) - \beta(f+\alpha T)}{(\mu+\beta T)^2} + \beta\left(\frac{f+\alpha T}{\mu+\beta T}\right)\left(\frac{\alpha(\mu+\beta T) - \beta(f+\alpha T)}{(\mu+\beta T)^2}\right.$$

$$\frac{\partial S}{\partial T}\bigg|_{f=0} = A + \frac{\alpha^2\mu}{(\mu+\beta T)^2} + \frac{\alpha^2\beta T\mu}{(\mu+\beta T)^3}$$

$$\boxed{C_F = AT + \frac{\alpha^2\mu T}{(\mu+\beta T)^2}\left(1 + \frac{\beta T}{\mu+\beta T}\right)}$$