M03Q.3 Angular Momentum | Chris Stanford

a) For $J = \frac{3}{2}$

$$\overset{j\ \ j_z}{\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle} = \overset{l\ \ l_z}{\left|1,1\right\rangle}\overset{s\ \ s_z}{\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle}$$

$$J_-\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle = (L_- + S_-)\left|1,1\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \qquad J_\pm\left|j,j_z\right\rangle = \hbar\sqrt{j(j+1) - j_z(j_z\pm1)}\left|j,j_z\pm1\right\rangle$$

$$\sqrt{\tfrac{3}{2}(\tfrac{5}{2}) - \tfrac{3}{2}(\tfrac{1}{2})}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \sqrt{1(2)-(1)(0)}\left|1,0\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{2}(\tfrac{3}{2}) - \tfrac{1}{2}(-\tfrac{1}{2})}\left|1,1\right\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle.$$

$$\sqrt{3}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \sqrt{2}\left|10\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \left|11\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|10\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \tfrac{1}{\sqrt{3}}\left|11\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle$$

$$J_-\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left(\sqrt{1(2)-0(-1)}\left|1-1\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{2}(\tfrac{3}{2}) - \tfrac{1}{2}(-\tfrac{1}{2})}\left|1,0\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle\right) + \tfrac{1}{\sqrt{3}}\sqrt{1(2)-1(0)}\left|10\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle$$

$$\sqrt{\tfrac{3}{2}(\tfrac{5}{2}) - (\tfrac{1}{2})(-\tfrac{1}{2})}\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left(\sqrt{2}\left|1-1\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \left|10\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle\right) + \tfrac{\sqrt{2}}{\sqrt{3}}\left|10\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle$$

$$2\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = \tfrac{2}{\sqrt{3}}\left|1-1\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \tfrac{2\sqrt{2}}{\sqrt{3}}\left|10\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}\left|1-1\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|10\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle = \left|1-1\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle.$$

For $J = \frac{1}{2}$

We form $\overset{j\ \ j_z}{\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle}$ by looking for something orthogonal to $\overset{j\ \ m_j}{\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle}$

$$\overset{j\ \ j_z}{\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle} = \tfrac{1}{\sqrt{3}}\left|10\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|11\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle \qquad \text{by inspection.}$$

$$J_-\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}\left(\sqrt{2}\left|1-1\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \left|10\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle\right) - \sqrt{\tfrac{2}{3}}\sqrt{2}\left|10\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|1-1\right\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle - \tfrac{1}{\sqrt{3}}\left|10\right\rangle\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle.$$

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$$\left.\begin{aligned}
\overset{j\ \ j_z}{\left\langle\tfrac{3}{2}\tfrac{3}{2}\right|}S_z\overset{j\ \ j_z}{\left|\tfrac{3}{2}\tfrac{3}{2}\right\rangle} &= \tfrac{1}{2}\\
\left\langle\tfrac{3}{2}\tfrac{1}{2}\right|S_z\left|\tfrac{3}{2}\tfrac{1}{2}\right\rangle &= \tfrac{2}{3}(\tfrac{1}{2}) + \tfrac{1}{3}(-\tfrac{1}{2}) = \tfrac{1}{6}\\
\left\langle\tfrac{3}{2}-\tfrac{1}{2}\right|S_z\left|\tfrac{3}{2}-\tfrac{1}{2}\right\rangle &= \tfrac{1}{3}(\tfrac{1}{2}) - \tfrac{2}{3}(\tfrac{1}{2}) = -\tfrac{1}{6}\\
\left\langle\tfrac{3}{2}-\tfrac{3}{2}\right|S_z\left|\tfrac{3}{2}-\tfrac{3}{2}\right\rangle &= -\tfrac{1}{2}
\end{aligned}\right\} = j_z/3 \qquad \therefore\ \delta E_{J,J_z} = \underbrace{-\frac{e\hbar B}{2m_e c}\left(1\pm\tfrac{1}{3}\right)}_{g_J}J_z$$

$$\left.\begin{aligned}
\left\langle\tfrac{1}{2}\tfrac{1}{2}\right|S_z\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle &= \tfrac{1}{3}(\tfrac{1}{2}) + \tfrac{2}{3}(-\tfrac{1}{2}) = -\tfrac{1}{6}\\
\left\langle\tfrac{1}{2}-\tfrac{1}{2}\right|S_z\left|\tfrac{1}{2}-\tfrac{1}{2}\right\rangle &= \tfrac{2}{3}(\tfrac{1}{2}) + \tfrac{1}{3}(-\tfrac{1}{2}) = \tfrac{1}{6}
\end{aligned}\right\} = -j_z/3$$

$$g_J = \begin{cases} -\frac{e\hbar B}{2m_e c}\left(1+\frac{1}{3}\right) & \text{for } J = \frac{3}{2}\\ -\frac{e\hbar B}{2m_e c}\left(1-\frac{1}{3}\right) & \text{for } J = -\frac{3}{2}\end{cases}$$

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b) For $J = l + \frac{1}{2}$ there is always the state:

$$\textcircled{1}\ |\overset{j}{l+\tfrac{1}{2}}, \overset{j_z}{l+\tfrac{1}{2}}\rangle = |\overset{l}{l}, \overset{l_z}{l}\rangle |\overset{s}{\tfrac{1}{2}}, \overset{s_z}{\tfrac{1}{2}}\rangle$$

$$J_-|l+\tfrac{1}{2}, l+\tfrac{1}{2}\rangle = (L_- + S_-)|l, l\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$\sqrt{2l+1}\,|l+\tfrac{1}{2}, l-\tfrac{1}{2}\rangle = \sqrt{2l}\,|l, l-1\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle + |l, l\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$|l+\tfrac{1}{2}, l-\tfrac{1}{2}\rangle = \sqrt{\frac{2l}{2l+1}}\,|l, l-1\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle + \frac{1}{\sqrt{2l+1}}|l, l\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

We find state $|l-\frac{1}{2}, l-\frac{1}{2}\rangle$ in the $j = l-\frac{1}{2}$ multiplet by finding a state that is orthogonal to the state $|l+\frac{1}{2}, l-\frac{1}{2}\rangle$.

$$\textcircled{2}\ |l-\tfrac{1}{2}, l-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2l+1}}|l, l-1\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle - \sqrt{\frac{2l}{2l+1}}|l, l-1\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

Now we consider $\langle S_z\rangle$ on the states ① and ②.

For $J = l+\frac{1}{2}$ $\quad \langle \overset{j}{l+\tfrac{1}{2}}, \overset{j_z}{l+\tfrac{1}{2}}|S_z|\overset{j}{l+\tfrac{1}{2}}, \overset{j_z}{l+\tfrac{1}{2}}\rangle = \frac{1}{2}(1) = \frac{1}{2}\frac{l+\frac{1}{2}}{l+\frac{1}{2}} = \frac{1}{2}\frac{j_z}{l+\frac{1}{2}} = \frac{j_z}{2l+1}$

For $J = l-\frac{1}{2}$ $\quad \langle l-\tfrac{1}{2}, l-\tfrac{1}{2}|S_z|l-\tfrac{1}{2}, l-\tfrac{1}{2}\rangle = \frac{1}{2l+1}\left(\frac{1}{2}\right) - \frac{2l}{2l+1}\left(\frac{1}{2}\right) = \frac{\frac{1}{2}-l}{2l+1} = \frac{-j_z}{2l+1}$

$$\therefore\ g_J = \begin{cases} -\frac{e\hbar B}{2m_e c}\left(1 + \frac{j_z}{2l+1}\right) & \text{for } J = l+\frac{1}{2} \\ -\frac{e\hbar B}{2m_e c}\left(1 - \frac{j_z}{2l+1}\right) & \text{for } J = l-\frac{1}{2} \end{cases}$$

c) Simly apply the theorem for $J = l+\frac{1}{2}$, $M = M' = l+\frac{1}{2}$, $\vec{A} = \vec{S}$

$$\langle l+\tfrac{1}{2}, l+\tfrac{1}{2}|S_z|l+\tfrac{1}{2}, l+\tfrac{1}{2}\rangle = \frac{\langle l+\frac{1}{2}|\vec{J}\cdot\vec{S}|l+\frac{1}{2}\rangle}{(l+\frac{1}{2})(l+\frac{3}{2})}\langle l+\tfrac{1}{2}, l+\tfrac{1}{2}|J_x + J_y + J_z|l+\tfrac{1}{2}\rangle(l+\tfrac{1}{2}\rangle$$

$$= \frac{\langle l+\frac{1}{2}|(\vec{L}+\vec{S})\cdot\vec{S}|l+\frac{1}{2}\rangle}{(l+\frac{1}{2})(l+\frac{3}{2})}\langle l+\tfrac{1}{2}, l+\tfrac{1}{2}|\cancelto{0}{\frac{J_+ + J_-}{2}} + \cancelto{0}{\frac{J_+ - J_-}{2i}} + J_z|l+\tfrac{1}{2}, l+\tfrac{1}{2}\rangle$$

$$= \frac{\langle l+\frac{1}{2}|(\vec{L}\cdot\vec{S}) + S^2|l+\frac{1}{2}\rangle}{\cancel{(l+\frac{1}{2})}(l+\frac{3}{2})}\cancel{(l+\tfrac{1}{2})}$$

$$= \frac{1}{l+\frac{3}{2}}\langle l+\tfrac{1}{2}|\tfrac{1}{2}((L+S)^2 - L^2 - S^2) + S^2|l+\tfrac{1}{2}\rangle$$

$$= \frac{1}{l+\frac{3}{2}}\frac{1}{2}\langle l+\tfrac{1}{2}|J^2 - L^2 + S^2|l+\tfrac{1}{2}\rangle$$

$$= \frac{1}{l+\frac{3}{2}}\frac{1}{2}\left[(l+\tfrac{1}{2})(l+\tfrac{3}{2}) - l(l+1) + \tfrac{1}{2}(\tfrac{3}{2})\right]$$

$$= \frac{1}{l+\frac{3}{2}}\frac{1}{2}\left[\cancel{l^2} + 2l + \tfrac{3}{4} - \cancel{l^2} - l + \tfrac{3}{4}\right]$$

$$= \frac{1}{\cancel{l+\frac{3}{2}}}\frac{1}{2}\left[\cancel{l+\tfrac{3}{2}}\right]$$

$$= \frac{1}{2}$$

Which is the same answer as part (b).