1 May 2003, Quantum Mechanics, Problem 2

1.1 (a)
Let $\epsilon$ be the absolute value of the energy. We assume that by bound state they mean something that oscillates inside the well, so that $V_0 > \epsilon$. Then the radial part of Schrödinger’s equation reads, for $l = 0$:

$$\frac{1}{2m} \frac{\partial^2 u}{\partial r^2} + V_0 u \Theta(r_0 - r) = \epsilon u$$

Let $k = \sqrt{2m(\epsilon - V_0)}$ and $p = \sqrt{2mE}$. Then the solutions look like:

$$u_{in} = \alpha \sin(kr) \quad r < r_0$$
$$u_{out} = D e^{-pr} \quad r > r_0$$

Both the functions and their derivatives must be continuous at $r = r_0$, so we get two equations:

$$\alpha \sin(kr_0) = D e^{-pr_0}$$
$$k\alpha \cos(kr_0) = -pD e^{-pr_0}$$
$$\tan(kr_0) = -\frac{k}{p} = -\frac{k}{\sqrt{2mV_0 - k^2}}$$

The first branch of the tangent goes to infinity at $kr_0 = \pi/2$. The function on the right-side is equal to 0 at $k = 0$ and decrease as $k$ increases, eventually going to $-\infty$ as $k$ approaches $\sqrt{2mV_0}$. Therefore, the two graphs will not cross each other if the function on the right goes to $-\infty$ before the new branch of the tangent appears, i.e., the critical value of the potential is given by:

$$\frac{\pi}{2r_0} = \sqrt{2mV_{crit}} \rightarrow V_{crit} = \left(\frac{\pi}{2r_0}\right)^2 \frac{1}{2m} \quad (1)$$

1.2 (b)
Let $E$ be the energy of the incoming particle, and let $k = \sqrt{2mE}$ and $p = \sqrt{2m(E + V_0)}$. The solution outside the potential region can be written (see [1]) as:

$$\psi = \frac{D}{2ikr} \left(e^{i(kr_0 + 2\delta_0)} - e^{-ikr_0}\right)$$

We must this solution to our interior solution found above:

$$\beta \sin(kr_0) = \frac{D}{2ik} \left(e^{i(kr_0 + 2\delta_0)} - e^{-ikr_0}\right) \quad \text{Continuity}$$
$$k\beta \cos(kr_0) = \frac{D}{2} \left(e^{i(kr_0 + 2\delta_0)} + e^{-ikr_0}\right) \quad \text{Derivative}$$
$$\frac{1}{p} \tan(pa) = \frac{1}{k} \tan(ka + \delta_0) \quad (2)$$

Using $p^2 = k^2 + 2mV_0$ and taking the limit $k \rightarrow 0$, we can get:
\[ \delta_0 = k \left[ \frac{\tan(\sqrt{2mV_0r_0})}{\sqrt{2mV_0}} - r_0 \right] \]  
\[ A = \frac{\tan(\sqrt{2mV_0r_0})}{\sqrt{2mV_0}} - r_0 \]  

1.3 (c)

\[ \lim_{V_0 \to 0} \frac{\tan(\sqrt{2mV_0r_0})}{\sqrt{2mV_0}} = r_0 \]
\[ \to \lim_{V_0 \to 0} A = 0 \]  

\[ \lim_{V_0 \to V_{crit}} \frac{\tan(\sqrt{2mV_0r_0})}{\sqrt{2mV_0}} = \lim_{x \to \pi/2^{-}} \tan(x) = \infty \]
\[ \to \lim_{V_0 \to V_{crit}} A = \infty \]  

1.4 (d)

We can write the exterior solution (see [1]), for \( l = 0 \), as:

\[ \psi = D \left[ \frac{\sin(kr)}{kr} + f \frac{e^{ikr}}{r} \right] \]

And the differential cross section is the absolute square of \( f \). Therefore, we need to match this version of the solution to the other one:

\[ D \left[ \frac{\sin(kr)}{kr} + f \frac{e^{ikr}}{r} \right] = \frac{D}{2ikr} (e^{i(kr+2\delta_0)} - e^{-ikr}) \]
\[ \frac{1}{2ik} + f = \frac{e^{i2\delta_0}}{2ik} \to f = \frac{e^{i2\delta_0}}{2ik} - 1 \]
\[ f = A \to \frac{d\sigma}{d\Omega} = |A|^2 \]
\[ \sigma = 4\pi|A|^2 = 4\pi \left[ \frac{\tan(\sqrt{2mV_0r_0})}{\sqrt{2mV_0}} - r_0 \right]^2 \]  

Notice that as \( V_0 \to V_{crit} \), the first term approaches \(-\infty\), so the cross section approaches \( \infty \). There is, thus, a pole at \( E=0 \).

References