M03Q.2 Scattering from a Central Potential

This problem deals with s-wave ($l = 0$) scattering of a particle of mass m from an attractive square well potential $V = -V_0\Theta(r-b)$

(a)

We define the following parameters $k \equiv \sqrt{2mE}/\hbar$, $\gamma \equiv \sqrt{2mV_0}/\hbar$ and $q \equiv \sqrt{k^2+\gamma^2}$.

The wave function can be written as $\psi(r) = \frac{u(r)}{r}$, where $u(r)$ solves the Schrodinger equation. In the region $r < b$,

$$-u'' - \gamma^2 u = k^2 u \tag{1}$$

which gives the solution $u(r) = A\sin qr$ for $r < b$.

Similarly, $u(r) = B\sin kr + \delta_0$ in the region $r > b$.

Since we are interested in bound states, we take the limit $k \to 0$.

The scattering length,

$$a = b - \frac{u'(b)}{u(b)} = b - \frac{\tan\gamma b}{\gamma} \tag{2}$$

in this limit as $q \to \gamma$.

In the presence of a bound state, the scattering length diverges, i.e. $\gamma b = \pi/2$, making it apparent that there is a critical value of V_0 such that a bound state arises, given by

$$V_{crit} = \left(\frac{\pi^2}{2b}\right)^2 \frac{\hbar^2}{2m} \tag{3}$$

(b)

To determine the phase shift, we match boundary conditions of $u(r)$ and $u'(r)$ at $r = b$. Dividing one equation by the other, we get the following condition

$$\frac{\tan qb}{q} = \frac{\tan kb + \delta_0}{k} \tag{4}$$

In the limit $k \to 0$, we approximate $\tan kb + \delta_0 \approx kb + \delta_0$. Solving for the phase shift, we get

$$\delta_0 = k\left(\frac{\tan qb}{q} - b\right) \tag{5}$$

Thus, we can define A to be

$$A \equiv \frac{\tan qb}{q} - b \tag{6}$$

(c) As $V_0 \to 0$, $q \to k$ so $A \to 0$.
As $V_0 \to V_{crit}^-$, $qb \to gammab \to \pi/2$ from the argument in part (a), thus A blows up.

(d)
The total cross section σ is given by

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0 \tag{7}$$

In the small k limit, $\sin^2\delta_0 \approx \delta_0^2$

$$\sigma \approx 4\pi\left(\frac{\tan\gamma b}{\gamma} - b\right)^2 \tag{8}$$

As $V_0 \to V_{crit}$, the cross section blows up, indicating that there is a pole at $E = 0$.

One thought on "M03Q.2 Scattering from a Central Potential"

Your solution is generally correct, but I have some remarks.
Some points require more explanation. Formula (2) contains a small mistake and has to be explained.
Why does the scattering length diverge when the bound state appears? If you use this fact in (a), it definitely has to be justified. This is a smart simplification trick, but I really think that, as it follows from the remaining parts of the problem, you were probably supposed to derive your result in part (a) in a different way.
Indeed, the part (c) of the problem asks you to show that the scattering length blows up as V_0 approaches its critical value V_{crit} . But you already used this fact in (a) to derive your answer for V_{crit} . Doesn't this loop look disturbing to you?