1 May 2003, Quantum Mechanics, Problem 1

Throughout this problem I will set $\mu = 1$, then restore the units in the end.

1.1 (a)

We start by pointing out that the problem is written incorrectly. The *hat* state must read:

$$\hat{\psi} = exp\left(i\phi(t)\frac{\sigma_z}{2}\right)\psi \tag{1}$$

$$\dot{\hat{\psi}} = \frac{i\dot{\phi}\sigma_z}{2}\hat{\psi} + e^{i\phi\sigma_z/2}\dot{\psi} = \frac{i\dot{\phi}\sigma_z}{2}\hat{\psi} - ie^{i\phi\sigma_z/2}H\psi$$

$$i\dot{\hat{\psi}} = -\frac{\dot{\phi}\sigma_z}{2}\hat{\psi} + e^{i\phi\sigma_z/2}He^{-i\phi\sigma_z/2}\hat{\psi}$$

This defines:

$$H_{rot} = -\frac{\dot{\phi}\sigma_z}{2} + e^{i\phi\sigma_z/2}He^{-i\phi\sigma_z/2}$$

Exploiting the fact $\sigma_z^2 = 1$ and expanding the exponential, we can prove:

$$e^{i\phi\sigma_z/2} = cos(\phi/2) + i\sigma_z sin(\phi/2) = \begin{pmatrix} cos(\phi/2) & 0 \\ 0 & cos(\phi/2) \end{pmatrix} + i\begin{pmatrix} sin(\phi/2) & 0 \\ 0 & -sin(\phi/2) \end{pmatrix} =$$

$$= \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$$

$$H = \begin{pmatrix} B_0 & B_1e^{-i\phi} \\ B_1e^{i\phi} & -B_0 \end{pmatrix}$$

$$e^{i\phi\sigma_z/2}He^{-i\phi\sigma_z/2} = \begin{pmatrix} B_0 & B_1 \\ B_1 & -B_0 \end{pmatrix}$$

When ϕ is linear, of the form $\omega_1 t + \gamma$, we have:

$$H_{rot} = \begin{pmatrix} -\omega_1/2 & 0 \\ 0 & \omega_1/2 \end{pmatrix} + \begin{pmatrix} B_0 & B_1 \\ B_1 & -B_0 \end{pmatrix} = \begin{pmatrix} -\omega_1/2 + B_0 & B_1 \\ B_1 & \omega_1/2 - B_0 \end{pmatrix} \tag{2}$$

And so this Hamiltonian is time-independent.

1.2 (b)

We have the new Schrödinger equation:

$$H_{rot}\hat{\psi} = \lambda\hat{\psi}$$

The eigenstates and eigenvalues are:

$$\hat{\psi}_+ = \begin{pmatrix} (-\omega_1/2 + B_0 + \omega')/B_1 \\ 1 \end{pmatrix} \quad \lambda = \omega'$$

$$\hat{\psi}_- = \begin{pmatrix} 1 \\ -(-\omega_1/2 + B_0 + \omega')/B_1 \end{pmatrix} \quad \lambda = -\omega'$$

$$\omega' \equiv \sqrt{\left(\frac{\omega_1}{2} - B_0\right)^2 + B_1^2}$$

Meanwhile, we have to express the initial state:

$$\hat{\psi}(-T) = \begin{pmatrix} 0 \\ e^{-i(-\omega_1 T+\gamma)/2} \end{pmatrix} = \frac{e^{-i(-\omega_1 T+\gamma)/2}}{1 + \frac{\left(-\frac{\omega_1}{2}+B_0+\omega'\right)^2}{B_1^2}} \left[\hat{\psi}_+ - \frac{\left(-\frac{\omega_1}{2} + B_0 + \omega'\right)}{B_1}\hat{\psi}_-\right]$$

$$\hat{\psi}(t) = \frac{e^{-i(-\omega_1 T+\gamma)/2}}{1 + \frac{\left(-\frac{\omega_1}{2}+B_0+\omega'\right)^2}{B_1^2}} \left[\hat{\psi}_+ e^{-i\omega'(t+T)} - \frac{\left(-\frac{\omega_1}{2} + B_0 + \omega'\right)}{B_1}\hat{\psi}_- e^{i\omega'(t+T)}\right]$$

$$\hat{\psi}(T) = \frac{e^{-i(-\omega_1 T+\gamma)/2}}{1 + \frac{\left(-\frac{\omega_1}{2}+B_0+\omega'\right)^2}{B_1^2}} \left[\hat{\psi}_+ e^{-2i\omega' T} - \frac{\left(-\frac{\omega_1}{2} + B_0 + \omega'\right)}{B_1}\hat{\psi}_- e^{2i\omega' T}\right]$$

We want the final state to be up, that is, we want:

$$\hat{\psi}(T) = \begin{pmatrix} e^{i(\omega_1 T+\gamma)/2} \\ 0 \end{pmatrix}$$

This gives us two equations:

$$\frac{e^{-i(-\omega_1 T+\gamma)/2}}{1 + \frac{\left(-\frac{\omega_1}{2}+B_0+\omega'\right)^2}{B_1^2}} \left[(-\omega_1/2 + B_0 + \omega')/B_1 e^{-2i\omega' T} - \frac{\left(-\frac{\omega_1}{2} + B_0 + \omega'\right)}{B_1} e^{2i\omega' T}\right] = e^{i(\omega_1 T+\gamma)/2}$$

$$\frac{e^{-i(-\omega_1 T+\gamma)/2}}{1 + \frac{\left(-\frac{\omega_1}{2}+B_0+\omega'\right)^2}{B_1^2}} \left[e^{-2i\omega' T} + \frac{\left(-\frac{\omega_1}{2} + B_0 + \omega'\right)}{B_1}(-\omega_1/2 + B_0 + \omega')/B_1 e^{2i\omega' T}\right] = 0$$

$$\frac{B_1}{B_1^2 + \left(-\frac{\omega_1}{2} + B_0 + \omega'\right)^2}(-\omega_1/2 + B_0 + \omega')\left[-2isin(2\omega' T)\right] = e^{i\gamma}$$

$$B_1^2 e^{-2i\omega' T} + \left(-\frac{\omega_1}{2} + B_0 + \omega'\right)^2 e^{2i\omega' T} = 0$$

The second equation becomes two:

$$B_1^2 cos(2\omega' T) + \left(-\frac{\omega_1}{2} + B_0 + \omega'\right)^2 cos(2\omega' T) = 0$$
$$-B_1^2 sin(2\omega' T) + \left(-\frac{\omega_1}{2} + B_0 + \omega'\right)^2 sin(2\omega' T) = 0$$

These require:

$$cos(2\omega' T) = 0 \rightarrow 2\omega' T = (2n+1)\pi/2$$
$$\left(-\frac{\omega_1}{2} + B_0 + \omega'\right)^2 = \left(-\frac{\omega_1}{2} + B_0\right)^2 + \left(\frac{\omega_1}{2} - B_0\right)^2 + B_1^2 + 2\left(-\frac{\omega_1}{2} + B_0\right)\omega' = B_1^2$$
$$\left(\frac{\omega_1}{2} - B_0\right)^2 = \left(\frac{\omega_1}{2} - B_0\right)\omega'$$

Assuming $B_1 \neq 0$, this requires:

$$\omega_1 = 2B_0 \quad (3)$$
$$\omega' = B_1$$
$$T = \frac{(2n+1)\pi}{4B_1} \quad (4)$$

The other equation we had to satisfy turns into:

$$\frac{B_1}{2B_1^2} B_1[-2isin((2n+1)\pi/2)] = e^{i\gamma}$$

This is also satisfied as long as we choose:

$$\gamma = -(2n+1)\pi/2$$

1.3 (c)

If you got this problem in the prelim, there's no way that you could have completed the first two parts, and still make it to the last one, so let's assume it's too hard. No, now seriously, it's just too long. The idea of this part is to solve the eigenvalue problem at some time t_0 and then plug in the time-dependent frequency in the energy levels. For the case above, we would have:

$$\lambda_+ = \sqrt{\left(\frac{\alpha t}{2} + B_0\right)^2 + B_1^2}$$
$$\lambda_- = -\sqrt{\left(\frac{\alpha t}{2} + B_0\right)^2 + B_1^2}$$

Then one has to take the same problem we took before and plug in the new frequency, then take the limit $T \rightarrow \infty$, hoping to obtain, in the end, the equivalent of the "up" state in the interaction picture.