M03M.3

Solution to M03M.3 —

We neglect effects of viscosity. We will use Navier Stokes: $\rho(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v})=-\vec{\nabla}p$; and continuity: $\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0$.

a) Let us write $p=p_0+p_1$, $\rho=\rho_0+\rho_1$, where the subscript 0 refers to the equilibrium state and the subscript 1 refers to a small perturbation. At equilibrium, the velocity profile is zero, and we let $\vec{v}$ denote a small perturbation.

The Navier-Stokes equation at equilibrium gives $\vec{\nabla}p_0=0$. Adding a small perturbation gives
$(\rho_0+\rho_1)\frac{\partial\vec{v}}{\partial t}=-\vec{\nabla}(p_0+p_1)=-\vec{\nabla}p_1$ (1)

where we have dropped terms that are second or higher order in perturbation. The $\vec{\nabla}p_0$ term vanishes due to the equilibrium condition.

Now the continuity equation gives
$\frac{\partial(\rho_0+\rho_1)}{\partial t}+\vec{\nabla}\cdot((\rho_0+\rho_1)\vec{v})=\frac{\partial\rho_1}{\partial t}+\vec{\nabla}\cdot((\rho_0+\rho_1)\vec{v})=0$ (2)
where the first ρ_0 term vanishes since ρ_0 is static at equilibrium.

Now we take the divergence of (1) and substitute (2). This gives
$\frac{\partial^2\rho_1}{\partial t^2}=\nabla^2 p_1$ (3)

From the ideal gas law we find that for constant temperature p and ρ are proportional. We will let $p=c_s^2\rho$ for some constant c_s. We will later interpret c_s as the speed of sound.

At equilibrium, we have $p_0=c_s^2\rho_0$ (4). After adding a small perturbation, we get $p_0+p_1=c_s^2(\rho_0+\rho_1)$.

Subtracting the two equations we get $p_1 = c_s^2 \rho_1$ (5). Substituting this into (3) gives $\frac{\partial^2 p_1}{\partial t^2} = c_s^2 \nabla^2 p_1$, which is a wave equation with speed c_s.

Substituting (4) & (5) into (1) gives
$\frac{\partial \vec{v}}{\partial t} = -\frac{c_s^2}{p_0+p_1}\vec{\nabla} p_1 \approx -\frac{c_s^2}{p_0}\vec{\nabla} p_1$
which allows us to infer the velocity profile from the gradient of the fluctuating pressure.

b) The boundary condition for pressure is $\hat{n}\cdot\vec{\nabla} p = 0$, where $\hat{n}$ is a unit vector normal to the boundary. Thus, if we put our system in a cubic box of side length L, we get
$p_1 = A\cos(\frac{n_1\pi}{L}x)\cos(\frac{n_2\pi}{L}y)\cos(\frac{n_3\pi}{L}z)e^{i\omega t}$
where $n_1, n_2, n_3 = 0, 1, 2, \ldots$. Substituting this into the pressure wave equation gives $\omega^2 = c_s^2 \frac{\pi^2}{L^2}(n_1^2 + n_2^2 + n_3^2)$. The mode $n_1 = n_2 = n_3 = 0$ gives $\omega = 0$ and $p_1 = A$, in which case the pressure fluctuation is constant. This simply shifts the equilibrium pressure p_0 by a constant, so we can neglect it.

c) We assume that $\rho(t)$ and $p(t)$ inside the box are uniform. We assume that the equilibrium pressure p_0 and density ρ_0 are uniform everywhere in the box, in the tube and outside. We assume that the velocity profile $u(t)$ in the tube is independent of position, and is positive for outward flow.

We first apply local mass conservation. The rate at which mass is flowing out of the box can be expressed in two ways:
$-\frac{d(\rho L^3)}{dt} = S\rho u$
$\frac{d\rho}{dt} = -\frac{uS}{L^3}\rho$
Since $p = c_s^2\rho$, we get
$\frac{dp}{dt} = -\frac{uS}{L^3}p$
Now set $p(t) = p_0 + p_1(t)$. Recalling that p_0 is static, and keeping only lowest order perturbation terms, we get:
$\frac{dp_1}{dt} = -\frac{Su}{L^3}p_0$ (6)
Notice that we treat u as a first order perturbation term.

Now, let P denote the pressure in the tube, and we will solve Navier-Stokes in the tube. Assume that the mass density in the tube is uniform and equal to the mass density $\rho(t)$ in the box. The velocity profile in the tube is $\vec{u} = u(t)\hat{x}$, where the $\hat{x}$-axis coincides with the axis of symmetry of the tube and points outward from the box. We set the origin at the center of the hole on the wall. Since $\vec{u}$ has no spatial dependence, the $\vec{u}\cdot\vec{\nabla}\vec{u}$ term in Navier-Stokes is zero. So we are left with
$\rho\frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} P$
Now apply perturbation theory and keep the lowest order terms:
$\rho_0\frac{\partial \vec{u}}{\partial t} = -\vec{\nabla}(P_0 + P_1) = -\vec{\nabla}(p_0 + P_1) = -\vec{\nabla} P_1$

where we assumed that the equilibrium pressure P_0 in the tube is the same as the equilibrium pressure p_0 in the box, which is uniform. Since the left hand side points along $\hat{x}$, therefore P only varies along $\hat{x}$. It follows that

$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x}$

Now integrate both sides with $\int_0^l dx$, keeping in mind that the left-hand side is independent of x:

$\rho_0 \frac{\partial u}{\partial t} l = -(p_0 - (p_0 + p_1)) = p_1$ (7) where for the pressure boundary conditions, we have assumed that $P(x=0) = p_0 + p_1$ and $P(x=l) = p_0$.

Equations (6) and (7) give us the coupled equations for $u(t)$ and $p(t)$. Taking d/dt of (6) and substituting (7) gives

$\frac{d^2 p_1}{dt^2} + \omega^2 p_1 = 0$ where $\omega = \sqrt{\frac{S p_0}{L^3 \rho_0 l}}$, which is the required angular frequency.

One thought on “M03M.3”

December 8, 2013 at 9:52 pm

OK, I generally agree with your solution.
Except of one point -- the process of sound wave propagation being isothermal. The changes in pressure are quite fast, so I'd rather assume that it is adiabatic. Then $c_s^2 = \frac{\partial p}{\partial \rho} = \gamma \frac{p}{\rho}$.
