M03M.3

Solution to M03M.3 — Helmholtz Resonator

a.

By dimensional analysis, the only combination of ρ_0 and p_0 with units of speed is $\sqrt{\frac{p_0}{\rho_0}}$, thus the wave equation for p_1 must be: $\frac{d^2 p_1}{dt^2} = \frac{p_0}{\rho_0} \nabla^2 p_1$. To get the velocity field, we could use the inviscid incompressible navier-stokes equation: $\rho(\frac{dv}{dt} + (v \cdot \nabla)v) = -\nabla p$ where we could relate pressure and density as above.

b.

At the edges of the box, we want Neumann boundary conditions, because the pressure pressure fluctuation presumably cant leave the box, thus solving the equation with these boundary conditions we have $p_1 = Ae^{iwt} \cos(2\pi mx/L) \cos(2\pi ny/L) \cos(2\pi lz/L)$ where A is some amplitude and m,n,l are integers. Putting this into the wave equation we get $w = \sqrt{\frac{p_0}{\rho_0}} (2\pi/L)\sqrt{n^2 + m^2 + l^2}$. There can't be a zero mode, because then the equilibrium pressure would change in time which is a contradiction.

c.

From the navier-stokes equation $\rho(\frac{dv}{dt} + (v \cdot \nabla)v) = -\nabla p$, since v has no spatial dependence and taking the density to lowest order, $(p(t) - p_{atm})/l = \rho_0 \frac{dv}{dt}$. From the integral version of the continuity equation for mass: $\int \frac{d\rho}{dt} dV + \int \rho v dA = \frac{d\rho}{dt} L^3 + \rho v S = 0$, thus differentiating this in terms of time and again taking the density to lowest order: $-\frac{d^2\rho}{dt^2} L^3/S = \rho_0 \frac{dv}{dt}$. Combining this with the previous equation, and letting $p'(t) = p(t) - p_{atm}$ we get: $-\frac{S p_0 p'(t)}{L^3 l \rho_0} = \frac{p_0}{\rho_0} \frac{d^2\rho}{dt^2} = \frac{d^2 p'(t)}{dt^2}$. Thus the frequency is: $\sqrt{\frac{Sp_0}{L^3 l \rho_0}}$

One thought on “M03M.3”

December 8, 2013 at 7:21 pm

Comment on part (a): dimensional analysis is definitely correct, but you can't get an exact coefficient this way. And actually there should be $\frac{\partial p}{\partial \rho}$, if you work more thoroughly through the wave equation derivation for the sound waves. Then it means that you can get $\frac{p}{\rho}$ only if you assume that sound wave is an isothermal process. However it makes more sense to assume that it is adiabatic, because it's quite fast.

Your part (b) looks correct (apart from the expression for the sound velocity taken from (a)).

Part (c) is OK too, except that you use again an isothermal approximation to relate p and ρ. I would say adiabatic approximation is more relevant, and it will change the answer by a dimensionless factor $\sqrt{\gamma}$.