M03M.3

Solution to M03M.3 — Helmholtz Resonator

a.firstly we can give the equation for conserved current and the equation for fluid dynamics(which allow one to infer the local velocity v of the air fluid from the gradient of the fluctuating pressure field) as

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (v\rho) \tag{1}$$

$$\frac{\partial (v\rho)}{\partial t} + \nabla \cdot (\rho v * sv) = -\nabla p \tag{2}$$

and we also have the relation of p and ρ given by thermal dynamics

$$pV^{\gamma} = Const, \frac{p}{\rho^{\gamma}} = Const \tag{3}$$

which gives the relation of perturbation of p and ρ

$$p_1 = \frac{p}{\rho}\gamma\rho_1 \tag{4}$$

here we have γ between 1(for constant temperature) and the isothermal parameter.

thus we have the wave equation for p and also exactly for ρ, by ignore the small term with two v in second equation

$$\frac{\partial^2 p}{\partial^2 t} - c_s^2 \nabla^2 p = 0, c_s^2 = \gamma \frac{p_0}{\rho_0} \tag{5}$$

b. a proper boundary condition should be $\partial_{verticle} p = 0$, combining the wave equation we have, this gives the form of solution we are looking for.

$$p_1 = A_k \cos w_k t + \phi \cos k_x x \cos k_y y \cos k_z z \tag{6}$$

$$w_k^2 = c_s^2 (k_x^2 + k_y^2 + k_z^2), k_x = \frac{2\pi n}{L}, k_y = \frac{2\pi m}{L}, k_z = \frac{2\pi n}{l} \tag{7}$$

where n, m, l are non-negative integers 0, 1, 2, 3..., and to have non-trivial fluctuation(which is part of the background p_0), we require n, m, l should not all be zero.

c. now we only consider the zero node. and we treat the deviation from the "balance" rather small, then in such approximation we have

$$L^3 \frac{d\rho_0}{dt} = -u\rho_0 S, \rho_0 l \frac{du}{dt} = p_0 - p_a \tag{8}$$

still note that we have relation between p and ρ (here we can take $\gamma = 1$)

$$dp = \frac{p}{\rho} d\rho \tag{9}$$

finally we get

$$\frac{d^2 d\rho}{dt^2} = -\frac{\rho_0 S}{p_0 L^3 l} d\rho \tag{10}$$

the related frequency is

$$\omega = \sqrt{\frac{\rho_0 S}{p_0 L^3 l}} \tag{11}$$