M03M.2

May 2003, Classical Mechanics , Problem 2

a) inclined plane

With the constraint expressed as $\frac{d\overrightarrow{R}}{dt} = a\overrightarrow{\omega} \times \overrightarrow{n}$, we can take further time drivative and get:

$$\frac{d^2\overrightarrow{R}}{dt^2} = a\frac{d\overrightarrow{\omega}}{dt} \times \overrightarrow{n}$$

We denote the constraint force to be $\overrightarrow{f}$ for friction which lies in the plan and $\overrightarrow{N}$ for supporting force which is perpendicular to the plane and pointing to the center of mass, which is parallel to the surface, and get these two equation:

$$Mg\overrightarrow{z} + \overrightarrow{f} + \overrightarrow{N} = M\frac{d^2\overrightarrow{R}}{dt^2}$$

$$\overrightarrow{f} \times a\overrightarrow{n} = I\frac{d\overrightarrow{\omega}}{dt}$$

We want to eliminate $\overrightarrow{f}$ and $\frac{d\overrightarrow{\omega}}{dt}$ in our equations, and keep $\frac{d^2\overrightarrow{R}}{dt^2}$. In order to do that,we may take the cross product of the equation we got from constraint and equation of motion with $\overrightarrow{n}$, we will get:

$$\frac{d^2\overrightarrow{R}}{dt^2} \times \overrightarrow{n} = a\left(\left(\frac{d\overrightarrow{\omega}}{dt} \cdot \overrightarrow{n}\right)\overrightarrow{n} - (\overrightarrow{n} \cdot \overrightarrow{n})\frac{d\overrightarrow{\omega}}{dt}\right)$$

$$Mg\,\overrightarrow{z} \times \overrightarrow{n} + \overrightarrow{f} \times \overrightarrow{n} = M\,\frac{d^2\,\overrightarrow{R}}{dt^2} \times \overrightarrow{n}$$

With $\frac{d\,\overrightarrow{\omega}}{dt} \cdot \overrightarrow{n} = 0$, we have:

$$-a\,\frac{d\,\overrightarrow{\omega}}{dt} = \frac{d^2\,\overrightarrow{R}}{dt^2} \times \overrightarrow{n}$$

Plug in $\overrightarrow{f} \times \overrightarrow{n}$ and $\frac{d\,\overrightarrow{\omega}}{dt}$, finally we get that:

$$Mg\,\overrightarrow{z} \times \overrightarrow{n} = \frac{7}{5}\,M\,\frac{d^2\,\overrightarrow{R}}{dt^2} \times \overrightarrow{n}$$

With $\frac{d^{\,2}\,\overrightarrow{R}}{dt^2}$ is perpendicular to $\overrightarrow{n}$ and $\overrightarrow{z}$ have angle θ with $\overrightarrow{n}$, then we will get:

$$\frac{d^2\,\overrightarrow{R}}{dt^2} = \frac{5}{7}g\sin(\theta)$$

b) rotating plane

Basically, in this case we have the same derivation only with constraint being changed into:

$$-\overrightarrow{\Omega} \times \overrightarrow{R} + \frac{d\overrightarrow{R}}{dt} = a\,\overrightarrow{\omega} \times \overrightarrow{n}$$

And in this case, the supporting force is canceled with the gravity, and we get the new equation:

$$\overrightarrow{f} = M\,\frac{d^2\,\overrightarrow{R}}{dt^2}$$

$$\overrightarrow{f} \times a\,\overrightarrow{n} = I\,\frac{d\,\overrightarrow{\omega}}{dt}$$

Then we can take the time derivative of the constraint equation and we get:

$$-\vec{\Omega}\times\frac{d\vec{R}}{dt}+\frac{d^2\vec{R}}{dt^2}=a\,\frac{d\vec{\omega}}{dt}\times\vec{n}$$

Take the cross product of angular momentum equation and we will get:($\vec{f}$ is parallel to the plane)

$$-a\,\vec{f}=I\,\frac{d\vec{\omega}}{dt}\times\vec{n}$$

By eliminating $\vec{f}$, we will simply get:

$$-\vec{\Omega}\times\frac{d\vec{R}}{dt}+\frac{7}{2}\,\frac{d^2\vec{R}}{dt^2}=0$$

Then we do a time integral of the equation we just got, we will get:

$$-\vec{\Omega}\times\vec{R}+\frac{7}{2}\,\frac{d\vec{R}}{dt}=\overrightarrow{const.}$$

So in the center of mass inertial frame, the sphere executes circular motion in the horizontal

with frequency $\frac{2}{7}\Omega$!

One thought on “M03M.2”

December 8, 2013 at 8:06 pm

Good, everything is correct.