M03M.2

Solution to M03M.2 — Rolling Sphere

Lets start by writing our rolling constraint,

$$\dot{\vec{r}} + a\vec{\omega} \times \hat{z} = 0 \tag{1}$$

Newton's 2^{nd} for the center of mass

$$m\ddot{\vec{r}} = -\nabla V + \vec{f} \tag{2}$$

and the torque about the center of mass

$$\tau = I\dot{\vec{\omega}} = -a\left(\hat{z} \times \vec{f}\right) \tag{3}$$

We differentiate the rolling constraint (1) and substitute in the torque equation (3) to find

$$\ddot{\vec{r}} = -a\left(\hat{z} \times \dot{\vec{\omega}}\right) = \frac{a^2}{I}\hat{z} \times \left(\hat{z} \times \vec{f}\right) = \frac{a^2}{I}\vec{f} \tag{4}$$

which we then substitute equation (2) into thereby eliminating the constraint force

$$m\ddot{\vec{r}} = \frac{-a^2}{I}\left(m\ddot{\vec{r}} + \nabla V\right) \rightarrow \left(m + \frac{I}{a^2}\right)\ddot{r} = -\nabla V \tag{5}$$

The form of our potential takes the standard

$$V = mgy\sin\theta \tag{6}$$

Separating the X and Y motions from equation (5), we find

$$\left(m + \frac{I}{a^2}\right)\ddot{x} = -\nabla V = 0 \tag{7}$$

$$\left(m+\frac{I}{a^2}\right)\ddot{y}=-\nabla V=mg\sin\theta \tag{8}$$

Which we note are equations for parabolic motion (constant velocity in x and constant acceleration in y). Plugging in the moment of inertia for a uniform sphere so generously provided by the examiners, we find

$$\left(m+\frac{I}{a^2}\right)\ddot{y}=mg\sin\theta\rightarrow\ddot{y}=\frac{5}{7}g\sin\theta \tag{9}$$

Part Deux

The sphere is now rolling on a level turntable which has constant angular velocity $\vec{\Omega}$ in the $\hat{z}$ direction. The modifications to our three equations from part a will be

$$\nabla V=0 \tag{10}$$

and

$$\dot{\vec{r}}+a\vec{\omega}\times\hat{z}=\vec{\Omega}\times\vec{r} \tag{11}$$

Again, differentiating our rolling constraint and substituting in our Newton's second for the center of mass we find

$$\ddot{\vec{r}}=\frac{-a^2}{I}f+\vec{\Omega}\times\dot{\vec{r}}=\frac{a^2}{I}m\ddot{\vec{r}}+\vec{\Omega}\times\dot{\vec{r}}\rightarrow\ddot{\vec{r}}=\frac{2}{7}\vec{\Omega}\times\dot{\vec{r}} \tag{12}$$

Integrating and rearranging the equation,

$$\dot{\vec{r}}-\frac{2}{7}\vec{\Omega}\times\vec{r}=Constant \tag{13}$$

which is the equation of motion for uniform circular motion in the inertial frame, with frequency $\frac{2}{7}\Omega$.

One thought on “M03M.2”

December 8, 2013 at 8:19 pm

What you've done looks correct.

Although after looking at some formulas I have a feeling that they may contain some typos.