M03M.2

Solution to M03M.2 — Rolling Sphere

1. The system of a sphere of radius a and mass M rolling down a plane of inclination θ is described by the following three equations:

$$\mathbf{R} = a\boldsymbol{\omega} \times \hat{\mathbf{n}} \tag{1}$$

$$M\ddot{\mathbf{R}} = Mg\sin\theta\hat{\mathbf{z}} + \mathbf{f} \tag{2}$$

$$I\dot{\boldsymbol{\omega}} = -a\hat{\mathbf{n}} \times \mathbf{f} \tag{3}$$

which represent the non-slip rolling conditions, force equation along the downhill $\hat{\mathbf{z}}$ direction and the torque equation about the center of mass.
Differentiating (1) and substituting $\dot{\omega}$ from (3),

$$\ddot{\mathbf{R}} = -\frac{a^2}{I}\mathbf{f} \tag{4}$$

Eliminate $\mathbf{f}$ from (3) to get

$$\ddot{\mathbf{R}}\left(1 + \frac{I}{Ma^2}\right) = g\sin\theta\hat{\mathbf{z}} \tag{5}$$

Substituting $I = \frac{2}{5}\,Ma^2$ for a sphere, we get

$$\ddot{\mathbf{R}} = \frac{5}{7} g \sin\theta \hat{\mathbf{z}} \tag{6}$$

which is the downhill acceleration which was required to be shown.

2. For the case of the plane rotating with angular velocity $\mathbf{\Omega}$, we have the new constraint condition

$$\dot{\mathbf{R}} = a\boldsymbol{\omega} \times \hat{\mathbf{n}} + \Omega \times \mathbf{R} \tag{7}$$

which comes from the relation between velocities in inertial and non-inertial frames, $\mathbf{v}_i = \mathbf{v}_n + \mathbf{\Omega} \times \mathbf{r}$. We also have the force and torque equations as before, neglecting gravity

$$M\ddot{\mathbf{R}} = \mathbf{f} \tag{8}$$

$$I\dot{\boldsymbol{\omega}} = -a(\hat{\mathbf{n}} \times \mathbf{f}) \tag{9}$$

Differentiating as before and eliminating $\mathbf{f}$ after substituting for $\dot{\omega}$,

$$\ddot{\mathbf{R}} = a\dot{\omega} \times \hat{n} + \mathbf{\Omega} \times \dot{\mathbf{R}} \tag{10}$$

$$\ddot{\mathbf{R}} = -\frac{Ma^2}{I}\ddot{\mathbf{R}} + \mathbf{\Omega} \times \dot{\mathbf{R}} \tag{11}$$

Substituting I for a sphere and rearranging, we get

$$\ddot{\mathbf{R}} = \frac{2}{7}\mathbf{\Omega} \times \dot{\mathbf{R}} = \frac{2}{7}\Omega\hat{\mathbf{n}} \times \dot{\mathbf{R}} \tag{12}$$

Which is the required equation for motion in a circle at frequency $\frac{2}{7}\Omega$.

One thought on “M03M.2”

December 8, 2013 at 8:07 pm

This is correct and clear.