M03M.2

Part A

We are provided with the equation for rolling without slipping (1), and may extract two more equations by using a Newtonian analysis to formulate the equation for linear motion of (2) and angular acceleration about (3) the center of mass:

$$\dot{\mathbf{R}} = a\boldsymbol{\omega} \times \hat{\mathbf{n}} \tag{1}$$

$$M\ddot{\mathbf{R}} = \mathbf{f} - \boldsymbol{\nabla} V = \mathbf{f} - mgsin(\theta)\hat{\mathbf{k}} \tag{2}$$

$$I\dot{\boldsymbol{\omega}} = (-a\hat{\mathbf{n}}) \times \mathbf{f} \tag{3}$$

Coordinates are defined such that the n-k-ω directions define a right-handed system. In this case, the gravitational force in the direction tangential to the surface was incorporated into (2), while $\mathbf{f}$ is the frictional force at the point of contact. From here, the problem is merely algebraic. (1) may be rewritten as:

$$\ddot{\mathbf{R}} = -\hat{\mathbf{n}} \times (a\boldsymbol{\omega}) \tag{4}$$

The expression for $\boldsymbol{\omega}$ in (3) may then be plugged into (4):

$$\mathbf{f} = -\frac{I}{a^2}\ddot{\mathbf{R}} \tag{5}$$

and as the problem statements tells us, we eliminate the force of constraint between (5) and (2):

$$\left(M + \frac{I}{a^2}\right)\ddot{\mathbf{R}} = \frac{7}{5}M\ddot{\mathbf{R}} = Mgsin(\theta)\hat{\mathbf{k}} \tag{6}$$

which is simply denotes the linear acceleration of $\frac{5}{7} gsin(\theta)$ along the downward direction that we were seeking.

Part B

For this case, the gravitational term in (2) goes away, while (3) remains the same and (1) is adjusted to a new form (7) that superimposes motion due to rotation of the ball with motion due to the spinning of the turntable. In this case, I simply define $\mathbf{R}$ as the vector connecting the axis of rotation to the ball's center of mass:

$$\dot{\mathbf{R}} = \Omega \hat{\mathbf{n}} \times \mathbf{R} + a\boldsymbol{\omega} \times \hat{\mathbf{n}} \tag{7}$$

This time, the only force on the ball is due to friction (2), which may be plugged directly into (3) to obtain:

$$\dot{\boldsymbol{\omega}} = -\frac{Ma}{I} \hat{\mathbf{n}} \times \ddot{\mathbf{R}} \tag{8}$$

The expression (8) for the angular acceleration may in turn be substituted into the time derivative of (7), leading to:

$$\ddot{\mathbf{R}} = \Omega \hat{\mathbf{n}} \times \dot{\mathbf{R}} + \frac{Ma^2}{I} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \ddot{\mathbf{R}}) = \Omega \hat{\mathbf{n}} \times \dot{\mathbf{R}} - \frac{Ma^2}{I} \ddot{\mathbf{R}} \tag{9}$$

This may be reduced into the form:

$$\ddot{\mathbf{R}} = \frac{2}{7} \Omega \hat{\mathbf{n}} \times \dot{\mathbf{R}} \tag{10}$$

which bears an obvious resemblance to the motion of a particle of mass m and charge q moving in a magnetic field of strength B:

$$\ddot{\mathbf{R}} = \frac{qB}{m} \hat{\mathbf{b}} \times \dot{\mathbf{R}} \tag{11}$$

where the prefactor to the cross product is the gyrofrequency of the particle. Since these same equations must bear the same solution, we know in essence that the ball on our turntable rolls in circles at a frequency of $\frac{2}{7}\Omega$, as expected.

One thought on “M03M.2”

October 26, 2013 at 9:16 pm

Very good.
I don't have any comments.
Just noticed two typos: in (4) and in the line which goes after it should be $\dot{\omega}$.