M03M.1

Solution to M03M.1

a) In the reference frame that rotate around the sun at angular frequency ω, the Sun and Earth are stationary. Because this is a non-inertial reference frame, we need an extra fictitious potential, $V_\omega = -m\omega^2 r^2/2$, here r is the distance from the Sun.

Then, the total potential energy can be written as:

$$V(r) = \frac{M_S m}{r} - \frac{m\omega^2 r^2}{2} + \frac{M_E m}{|r-R|}$$

In the L2 point, we have $\frac{\partial V(r)}{\partial r} = 0$, and r>R.

$$\frac{\partial V(r)}{\partial r} = \frac{GM_S m}{r^2} + \frac{GM_E m}{(r-R)^2} - m\omega^2 r = 0$$

That is : $GM_S(r-R)^2 - \omega^2 r^3 (r-R)^2 + GM_E r^2 = 0$

If we define d=r-R, which is the distance from the Earth.

Than the equation becomes: $GM_S d^2 - \omega^2 (d+R)^3 d^2 + GM_E (d+R)^2 = 0$

Because $\omega^2 = \frac{GM_S}{R^3}$

Then

$$M_S d^2 - \frac{M_S}{R^3}(d+R)^3 d^2 + M_E (d+R)^2 = 0$$

Define: $f(d) = M_S d^2 - \frac{M_S}{R^3}(d+R)^3 d^2 + M_E (d+R)^2$

We have $f(0)>0$ and $f(d)\to-\infty$ when $d\to\infty$. So the equation f(d)=0 must have least a solution at d>0, which means the L2 point does exist outside the orbit of Earth.

b)Put $\beta=\frac{M_E}{M_S} into the equation above,

$$d^2-\frac{1}{R^3}(d+R)^3d^2+\beta(d+R)^2=0$$

Assume $d=d_0+A\beta^l+higherorders$

Notice that when $\beta=0$, the equation becomes:

$$d_0^2-\frac{1}{R^3}(d_0+R)^3d_0^2=0$$

, which means $d_0=0$;

Now let $d=A\beta^l+o(\beta^l)$, put this into the equation and only keep the leading order of β, we have:

$$(A\beta^l+o(\beta^l))^2-\frac{1}{R^3}(A\beta^l+o(\beta^l)+R)^3(A\beta^l+o(\beta^l))^2+\beta(A\beta^l+o(\beta^l)+R)^2$$
$$=0$$

$$(A^2\beta^{2l}+o(\beta^{3l}))[-\frac{3A\beta^lR^2+o(\beta^{2l})}{R^3}]+\beta(2A\beta^lR+o(\beta^{2l})+R^2)=0$$

$$-\frac{3A^3\beta^{3l}}{R}+o(\beta^{4l})+2AR\beta^{l+1}+R^2\beta+o(\beta^{1+2l})=0$$

In this equation, the lowest order of β is β^{3l} or β. Only keep the β^{3l} and β terms, and we get:

$$-\frac{3A^3\beta^{3l}}{R}+R^2\beta=0$$

So we have $l=1/3$ and $A=\sqrt[3]{\frac{1}{3}}R$

i.e. $d=\sqrt[3]{\frac{\beta}{3}}R$

c)We have:

$$\frac{\partial V(r)}{\partial r}=\frac{GM_Sm}{r^2}+\frac{GM_Em}{(r-R)^2}-m\omega^2r=0$$

$$\frac{\partial^2 V(r)}{\partial r^2} = -2\,\frac{GM_S m}{r^3} - 2\,\frac{GM_E m}{(r-R)^3} + m\omega^2$$

Put $m\omega^2 = \frac{GM_S m}{r^3} + \frac{GM_E m}{(r-R)^2 r}$ into the equation.

$$\frac{\partial^2 V(r)}{\partial r^2} = -\,\frac{GM_S m}{r^3} - 2\,\frac{GM_E m}{(r-R)^3} + \frac{GM_E m}{(r-R)^2 r}$$

From previous result, $r = R + d << 2R$, so $\frac{\partial^2 V(r)}{\partial r^2} < 0$, which means L2 point is not a stable point.

One thought on "M03M.1"

October 27, 2013 at 12:15 am

Everything looks correct.
One typo: you have wrong signs for the gravitational terms in your expressions for $V(r)$.