M03M.1

Figures

Part a)

"F=ma" in the rotating frame (remembering the fictitious force)

$$m\ddot{x} = m\omega^2 x - \frac{GM_E m}{|x - x_E|^3}(x - x_E) - \frac{GM_S m}{|x|^3}x \tag{1}$$

where earth and sun masses are denoted by the subscripts E and S respectively. At the fixed point $\ddot{x} = \dot{x} = 0$ and thus:

$$\omega^2 x = \frac{GM_E}{|x - x_E|^3}(x - x_E) + \frac{GM_S}{|x|^3}x \tag{2}$$

The left and right hand sides are plotted in figure 2 and it can be seen there are three solutions with one outside of the earth's orbit.

Part b)

We are interested in the case where $x > x_E$ considering this and using the approximation that $\omega^3 \approx \frac{GM_S}{x_E^3}$ we have:

$$(\frac{x}{x_E})^3 = 1 + \frac{\beta x^2}{(x - x^E)^2} \tag{3}$$

with $\beta = \frac{M_E}{M_S}$. Now assuming that β is small:

$$\frac{x}{x_E} = (1 + \frac{\beta x^2}{(x - x^E)^2})^{\frac{1}{3}} = 1 + \frac{\beta x^2}{3(x - x^E)^2} + O(\beta^2) \tag{4}$$

$$(x - x_E)^3 = \frac{\beta x_E x^2}{3} \tag{5}$$

Now it is straight forward to see that if $\beta = 0$ then $x = x_E$. Using this we write $x = x_E + A\beta^k$ (where $k > 0$). We use this expansion in the right hand side and keep only the lowest order terms in β.

$$(x - x_E)^3 = \frac{\beta x_E^3}{3} \tag{6}$$

Thus: $x = x_E + x_E(\frac{\beta}{3})^{\frac{1}{3}}$

Using the values given shows that $x - x_E = 1.5 \times 10^6$ km.

Part c)

Returning to the equations of motion and writing x as $x = x_0 + \delta$, where δ is a small parameter and x_0 is the L2 point, we have:

$$m\ddot{\delta} = m\omega^2(x_0 + \delta) - \frac{GM_E m}{|x_0 + \delta - x_E|^3}(x_0 + \delta - x_E) - \frac{GM_S m}{|x_0 + \delta|^3}(x_0 + \delta) \tag{7}$$

As δ is small we can taylor expand the gravitational terms (and using $x - x_E$ and x are both positive):

$$\ddot{\delta} = \omega^2 x_0 - \frac{GM_E}{(x_0 - x_E)^2} - \frac{GM_S}{x_0^2} + (\omega^2 + \frac{2GM_E}{(x_0 - x_E)^3} + \frac{GM_S}{x_0^3})\delta \tag{8}$$

Now the first three terms cancel as x_0 is the L2 point. The term in the brackets is a positive constant thus: $\ddot{\delta} = \alpha^2\delta$ where $\alpha^2 = \omega^2 + \frac{2GM_E}{(x_0 - x_E)^3} + \frac{GM_S}{x_0^3}$
and so: $\delta = Ae^{\alpha t} + Be^{-\alpha t}$, where A and B are integration constants. This shows that the fixed point is unstable as small perturbations grow exponentially.

October 27, 2013 at 1:53 am

Solution is correct and all the answers are correct too.
However, I should note that equation (1) is, stricktly speaking, not correct. Equation of motion in the rotationg frame has extra terms due to rotation. The famous Coriolis force is missing from your equation. However, it is not important for this problem as, indeed, $\dot{x} = 0$, as you have mentioned.
Also there is a typo right before your formula (3) -- there whould be ω^3.