1 M02T.2

From the formula for non-relativistic chemical potential

$$\mu = T \log \frac{n}{g} \left(\frac{mT}{2\pi\hbar^2} \right)^{-\frac{3}{2}} + \epsilon_0, \tag{1}$$

where we units $c = 1$ and g is number of spin states ($g_e = g_p = 2$, $g_H = 4$) we get

$$n_i = g_i \left(\frac{m_i T}{2\pi\hbar^2} \right)^{\frac{3}{2}} e^{\frac{\mu - m_i}{T}}. \tag{2}$$

Since protons, electrons and hydrogen atoms are in equilibrium and the universe is neutral

$$\mu_H = \mu_e + \mu_p, \tag{3a}$$
$$n_p = n_e. \tag{3b}$$

Using formula for density in a combination to get chemical potential to vanish, we get

$$\frac{n_p^2}{n_H} = \frac{n_e n_p}{n_H} = \left(\frac{m_e T}{2\pi\hbar^2} \right)^{\frac{3}{2}} e^{-\frac{\Delta}{T}}, \tag{4}$$

where $\Delta = m_e + m_p - m_H = 13.6eV$ is binding energy and we assumed $m_p \approx m_H$. If at temperature $T_0 = 0.3eV$ half of protons were bound, that means that $n_p = n_e = n_H$ and hence

$$n_p = n_H = \left(\frac{m_e T_0}{2\pi\hbar^2} \right)^{\frac{3}{2}} e^{-\frac{\Delta}{T_0}} \approx 1 \cdot 10^7 \frac{1}{m^3} \tag{5}$$

Photon density is

$$n = g_\gamma \int \frac{1}{e^{\frac{pc}{T}} - 1} \frac{4\pi p^2 dp}{(2\pi\hbar)^3} = \frac{g_\gamma T^3}{2\pi^2 (\hbar c)^3} \int_0^\infty \frac{u^2 du}{e^u - 1} = \frac{2T^3}{\pi^2 (\hbar c)^3} \zeta(3) \approx 8.5 \cdot 10^{17} \frac{1}{m^3}. \tag{6}$$

Hence proton density is of order 10^{-10} smaller than photon density.