1 May 2002, Quantum Mechanics, Problem 3

1.1 (a)

In order to have the field be B_0 in the strap, and to have $\boldsymbol{A} = \boldsymbol{0}$ for $x < 0$, we need:

$$\boldsymbol{A} = B_0 x \hat{y} \quad 0 < x < d$$

Then the vector potential has to be continuous at $x = d$, so we must have:

$$\boldsymbol{A} = B_0 d \hat{y} \quad x > d$$

In order to ensure gauge covariance, we need to impose:

$$\boldsymbol{p} \rightarrow \boldsymbol{p} - q\boldsymbol{A}$$

This makes our Hamiltonian:

$$H = \frac{p_x^2}{2m} + \frac{(p_y - qA)^2}{2m} + \frac{p_z^2}{2m} \tag{1}$$

1.2 (b)

Using the vector potential calculated above, we get:

$$HTe^{i\tilde{k}x} = \left(\frac{\tilde{k}^2}{2m} + \frac{q^2 B^2 d^2}{2m} \right) Te^{i\tilde{k}x}$$

$$H(e^{ikx} + Re^{-ikx}) = \frac{k^2}{2m}(e^{ikx} + Re^{-ikx})$$

Since a magnetic field does no work, so energy must be conserved. Another argument is that the energy is either with the particle or in the field, and in both cases total energy is conserved. So:

$$\tilde{k} = \sqrt{k^2 - q^2 B_0^2 d^2} \tag{2}$$

1.3 (c)

When the argument of the square root becomes negative, $\tilde{k}$ becomes imaginary. This means that the incident energy is not high enough for the particle to make it through the barrier, and therefore the wavefunction on the other side of the barrier is a decaying exponential. The critical value of the energy is the one that makes $\tilde{k}$ equal 0:

$$E_0 = \frac{k_0^2}{2m} = \frac{(eB_0 d)^2}{2m} \tag{3}$$

Classically, the particle enters a circular orbit as soon as it enters the magnetic field region. The radius of that orbit is:

$$evB_0 = \frac{mv^2}{r} \rightarrow r = \frac{mv}{eB_0} = \frac{k}{eB_0}$$

If the length of the barrier is larger than the radius of the orbit, then the particle will turn around before reaching the end of the barrier and never make it to the other side. The critical value, then, is:

$$r = \frac{k}{eB_0} = d \rightarrow k = deB_0 \rightarrow E = \frac{(eB_0 d)^2}{2m} \tag{4}$$

1.4 (d)

The probability current is:

$$\boldsymbol{j} = \frac{i}{2m}(\psi \nabla \psi^* - \psi^* \nabla \psi)$$

But the gradients are now covariant gradients. Since we had $\boldsymbol{p} \rightarrow \boldsymbol{p} - q\boldsymbol{A}$, so:

$$-i\nabla \rightarrow -i\nabla - q\boldsymbol{A} = -i(\nabla - iq\boldsymbol{A})$$

$$\nabla_C =< \frac{\partial}{\partial x}, \frac{\partial}{\partial y} - iqA, \frac{\partial}{\partial z} >$$

$$\nabla_C \psi_{trans} = i(\tilde{k}\hat{x} - qB_0 d\hat{y})Te^{i\tilde{k}x}$$

$$\boldsymbol{j} = \frac{i}{2m}(|T|^2 i(\tilde{k}\hat{x} - qB_0 d\hat{y}) + |T|^2 i(\tilde{k}\hat{x} - qB_0 d\hat{y}) = -\frac{1}{m}|T|^2(\tilde{k}\hat{x} - qB_0 d\hat{y}) \tag{5}$$

1.5 (e)

As $d \rightarrow 0$, we can enforce continuity of the wavefunction and its derivative at $x = 0 = d$:

$$1 + R = T$$

$$ik(1 - R) = i\tilde{k}T$$

$$T = \frac{2k}{k + \tilde{k}} \tag{6}$$

$$R = \frac{k - \tilde{k}}{k + \tilde{k}} \tag{7}$$