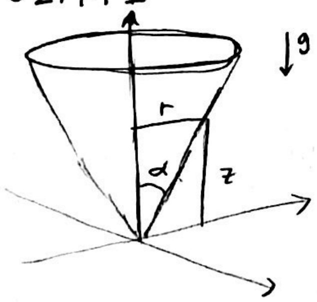
M02: M1



a) $\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz$

$z\tan\alpha = r \Rightarrow \dot{z}\tan\alpha = \dot{r}$

$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \cot^2\alpha\,\dot{r}^2) - mgr\cot\alpha$

$= \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2\right) - mgr\cot\alpha$

$\theta: \frac{d}{dt}(mr^2\dot{\theta}) = 0 \Rightarrow mr^2\dot{\theta} = L$ (Conservation of Ang. Mom.)

$\mathcal{L} = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2\right) - mgr\cot\alpha$

$r: \frac{d}{dt}\left(m\frac{\dot{r}}{\sin^2\alpha}\right) = \frac{m}{\sin^2\alpha}\ddot{r} = mr\dot{\theta}^2 - mg\cot\alpha$

$$\boxed{\begin{aligned} r&: \frac{m}{\sin^2\alpha}\ddot{r} = \frac{L^2}{mr^3} - mg\cot\alpha \\ \theta&: mr^2\dot{\theta} = L \end{aligned}}$$

b) $\ddot{r} = 0 \Leftarrow \dot{z} = 0$

$\frac{L^2}{mr_0^3} - mg\cot\alpha = 0$

$r_0 = z_0\tan\alpha \Rightarrow L^2 = m^2 g z_0^3 \tan^2\alpha = m^2 r_0^4 \dot{\theta}_0^2 \quad (\dot{\theta}_0 = \omega)$

$\dot{\theta}_0^2 = \frac{g}{z_0\tan^2\alpha} \Rightarrow \boxed{\omega = \sqrt{\frac{g}{z_0}}\cot\alpha}$

c) $r = r_0 + u \Rightarrow \ddot{r} = \ddot{u}$

$\frac{m}{\sin^2\alpha}\ddot{u} = \frac{L^2}{m(r_0+u)^3} - mg\cot\alpha$

Taylor Exp.

$\frac{m}{\sin^2\alpha}\ddot{u} = \frac{L^2}{m r_0^3} - \frac{3L^2}{mr_0^4}u - mg\cot\alpha$

$\frac{m}{\sin^2\alpha}\ddot{u} = -\frac{3L^2}{mr_0^4}u$

$\ddot{u} = -\frac{3L^2}{m^2r_0^4}\sin^2\alpha\, u$

Motion is stable.

$\Omega^2 = \frac{3L^2}{m^2r_0^4}\sin^2\alpha = \frac{3m^2r_0^4\omega^2}{m^2r_0^4}\sin^2\alpha$

$\left(\frac{\Omega}{\omega}\right)^2 = 3\sin^2\alpha$

$$\boxed{\Omega/\omega = \sqrt{3}\sin\alpha}$$