M02E.1 - Iron Ring with a Gap (Solution by Jim Wu)

N turns of a wire are wrapped around an iron ring in which a small gap has been cut. The radius of the ring is a and the width of the gap is w, with $w \ll a$. A current I flows in the wire. The magnetic permeability of the iron is μ.

(a) Find the B field in the gap.

(b) Find the force per unit area on the faces of the gap. Does the gap have the tendency to widen or contract?

Solution:

(a) From Ampere's law,

$$\int \mathbf{H} \cdot d\mathbf{l} = i_{free} \tag{1}$$

and since the $\mathbf{H}$ field goes around in a circle, this tells us that

$$H_{iron}(2\pi a - w) + H_{gap} w = NI \tag{2}$$

Assuming that iron is a linear medium, $\mathbf{B}_{iron} = \mu \mathbf{H}_{iron}$. In the gap, we have a similar relation $\mathbf{B}_{gap} = \mu_0 \mathbf{H}_{gap}$. Additionally, we know from $\nabla \cdot \mathbf{B} = 0$ that at the boundary between the iron and the gap, the normal component of the $\mathbf{B}$ field is continuous. This suggests that $B_{iron} = B_{gap} = B$.

So using the relations $H_{iron} = \frac{B}{\mu}$ and $H_{gap} = \frac{B}{\mu_0}$. Hence

$$\frac{B}{\mu}(2\pi a - w) + \frac{B}{\mu_0} w = NI \qquad \Rightarrow \qquad B = \frac{\mu_0 \mu N I}{\mu_0(2\pi a - w) + \mu w} \tag{3}$$

and since $w \ll a$, we can neglect the w in the parenthesis in (3), leaving us with

$$B \approx \frac{\mu_0 \mu N I}{2\pi \mu_0 a + \mu w} \tag{4}$$

(b) We can think about this in terms of the Maxwell stress-energy tensor. Since we have no electric fields anywhere, then in the gap, (if we set our coordinates such that z points upward through the gap), the stress-energy tensor is

$$T_{ij}^{gap} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}\delta_{ij}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \frac{1}{2\mu_0} B^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

while in the iron core, the stress tensor looks like

$$T_{ij}^{iron} = \frac{1}{2\mu} B^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If we take a thin box of arbitrarily small height ϵ and area A, then the force on upper face of the gap is given by

$$F_i = \oint T_{ij} da_j$$

The only contributions are in the z direction and since the elements of the tensor are constant, then

$$\frac{F_z}{A} = \frac{B^2}{2}\left(\frac{1}{\mu} - \frac{1}{\mu_0}\right)$$

where A is the area of the faces of the gap. Given that the magnetic permeability of iron is $\mu \gg \mu_0$, then the force is negative and we can approximate the force per unit area as

$$\frac{F_z}{A} \approx -\frac{1}{2\mu_0} B^2$$

This suggests that the gap has a tendency to contract. This makes sense as we can imagine the magnet having a north pole on one end and a south pole on the other, creating an attractive force.

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