1 May 2001, Thermodynamics, Problem 3

1.1 (a)

Suppose the field is in the y direction, then it can be written in terms of a vector potential:

$$\boldsymbol{A} = Bz\hat{x}$$

and we can write the Hamiltonian as usual, but with the gauge covariant derivative:

$$H = \frac{(p_x - eBz)^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$$

The particle is completely free to move in the y direction, but the x and z directions are intertwined. By assuming that the wavefunction doesn't depend on x, we can say that the z-dependence of the wavefunction is like a harmonic oscillator, of frequency $\omega = eB/m$, and so the energy levels are:

$$E_n = \hbar eB/m(n + 1/2) \tag{1}$$

About the degeneracies, it's not very clear to me how you can compute that. The Russians have a derivation but I don't understand it.

1.2 (b)

The definition of the grand partition function is:

$$\mathcal{Z} = \sum_{N=0}^{\infty} \sum_{\{n_p^i\}} \prod_p e^{-\beta n_p^i(\epsilon_p - \mu)}$$

which means:

(i) we sum over the total number of particles in the system;

(ii) for a given N', we sum over all the possible microstates that have N' particles;

(iii) each microstate is defined by the set $\{n_p^i\}$ of occupation numbers for each momentum p;

(iv) for each given microstate, we compute the product of the exponential terms generated by each momentum.

This was for the sake of understanding what the grand partition function means. There is a mathematical derivation (which you will just have to believe me), that we can write it as:

$$\mathcal{Z} = \prod_i \mathcal{Z}_i^{g_i}$$

where this means we compute the product of the grand partition functions for each momentum state i. The exponent g_i is the degeneracy of state i. So now we have to understand what $\mathcal{Z}_i$ means. Let's look back at the definition, and guess that it will be the same:

$$\mathcal{Z}_i = \sum_{N=0}^{\infty} \sum_{\{n_p^i\}} \prod_p e^{-\beta n_p^i(\epsilon_p - \mu)} \quad (wrong)$$

But now we can't be taking the product over all momenta, because we only want the partition function for one momentum, so we remove that product:

$$\mathcal{Z}_i = \sum_{N=0}^{\infty} \sum_{\{n_p^i\}} e^{-\beta n_p^i (\epsilon_p - \mu)} \quad (wrong)$$

But it doesn't make sense to be summing over total numbers of particles, because we are computing something that is valid only for one momentum state i, so we remove that too:

$$\mathcal{Z}_i = \sum_{\{n_p^i\}} e^{-\beta n_p^i (\epsilon_p - \mu)} \quad (right)$$

Now this is correct, as long as we interpret the sum to mean that we are summing over all possible values of the occupation number of the momentum state i. For fermions, each state of momentum p can only be occupied by 0 or 1 particles. Therefore, it's very easy to get the expression:

$$\mathcal{Z}_{i_{FD}} = 1 + e^{-\beta(\epsilon_{p_i} - \mu)}$$

If you are unhappy about this derivation, so am I. But at this point it's what gets me closest to understanding how to use grand partition functions. Now we introduce the activity and the grand potential:

$$\mathcal{Z}_{i_{FD}} = 1 + z e^{-\beta \epsilon_{p_i}}$$

$$\mathcal{Z} = \prod_i (1 + z e^{-\beta \epsilon_{p_i}})^{g_i} = \prod_n (1 + z e^{-\beta \hbar e B/m(n+1/2)})^{g_n} \tag{2}$$

which I cannot compute, at least until I know the degeneracy. The grand potential is:

$$\Phi_G = E - TS - \mu N = -kT ln \mathcal{Z} = -kT \sum_n g_n ln(1 + z e^{-\beta \hbar e B/m(n+1/2)})$$

$$d\Phi_G = -SdT - PdV + Nd\mu$$

$$P = -\left(\frac{\partial \Phi_G}{\partial V}\right)_{\mu,T} \tag{3}$$

Here I'm totally stuck, because I don't know what approximation to make for low density, and I don't know what the degeneracy is.