1 May 2001, Quantum Mechanics, Problem 1

1.1 (a)

Use the Pauli spin matrices to write the hamiltonian in matrix form:

$$\mathbf{H} = -\frac{\gamma B_0}{2}\begin{pmatrix} cos\theta & sin\theta e^{-i\phi} \\ sin\theta e^{i\phi} & -cos\theta \end{pmatrix}$$

The energy eigenvalues are those that solve the equation:

$$det(\mathbf{H} - E\mathbf{I}) = 0$$

where $\mathbf{I}$ is the two-dimensional identity matrix. The solutions turn out to be:

$$E = \pm\frac{\gamma B_0}{2}$$

and the normalized eigenstates are:

$$|\theta\,\phi>_+ = \begin{pmatrix} sin(\theta/2)e^{-i\phi} \\ -cos(\theta/2) \end{pmatrix}$$

$$|\theta\,\phi>_- = \begin{pmatrix} cos(\theta/2) \\ sin(\theta/2)e^{i\phi} \end{pmatrix} \tag{1}$$

Notice that the places where I put the exponentials seems arbitrary. After all, the wavefunctions can be multiplied by a unitary phase factor without changing the physics. However, I had to choose it like this so that when $\theta \to 0$, the eigenstates become independent of ϕ, for ϕ is not well-defined at $\theta = 0$.

1.2 (b)

The state will look like:

$$\Psi(t) = \alpha(t)\psi_{ground}(t) + \beta(t)\psi_{excited}(t)$$

where $\psi(t)$ are the eigenstates we found in part a, with ωt replacing ϕ. Plug this expression into the time-dependent Schrdinger's equation. You will get:

$$i\dot{\alpha}(t)\psi_g(t) + i\alpha(t)\dot{\psi_g}(t) + i\dot{\beta}\psi_e(t) + i\beta(t)\dot{\psi_e}(t) = \alpha(t)E_g\psi_g(t) + \beta(t)E_e\psi_e(t)$$

Apply $<\psi_g|$ to obtain:

$$i\dot{\alpha}(t) + i\alpha(t) <\psi_g|\dot{\psi_g}> + \beta(t) <\psi_g|\dot{\psi_e}> = \alpha(t)E_g(t)$$

If $\omega = 0$, then $\beta = 0$ for all time. Thus, we expect β to be of order ω. Also, we know that $\dot{\psi_e}$ is first order in ω. Therefore, the third term above is of second order and we can ignore it:

$$i\dot{\alpha(t)} + i\alpha(t) <\psi_g|\dot{\psi_g}> = \alpha(t)E_g(t)$$

This suggests a solution of the form $\alpha(t) = e^{i\varphi}$ (and the problem tells us it is, too!):

$$-\dot{\varphi} + i < \psi_g|\dot{\psi}_g >= E_g$$
$$\varphi = -E_g t - \phi(t) sin^2(\theta/2) = \frac{\gamma B_0 t - \omega t(1 - cos\theta)}{2} \tag{2}$$

A problem very similar to this one shows up in the book ”Princeton Problems in Physics”, and the result is very similar.

1.3 (c)

Return to the equation that arose from Schrdinger's equation:

$$i\dot{\alpha}(t)\psi_g(t) + i\alpha(t)\dot{\psi}_g(t) + i\dot{\beta}\psi_e(t) + i\beta(t)\dot{\psi}_e(t) = \alpha(t)E_g\psi_g(t) + \beta(t)E_e\psi_e(t)$$

and apply $< \psi_e|$, to obtain:

$$i\alpha(t) < \psi_e(t)|\dot{\psi}_g(t) > +i\dot{\beta} + i\beta(t) < \psi_e(t)|\dot{\psi}_e(t) >= \beta(t)E_e$$

Plug in $\alpha(t)$ as calculated in part b, and calculate the dot products from the eigenstates found in part a, with $\phi = \omega t$, to obtain:

$$e^{it[-E_g + \omega cos^2(\theta/2)]}\omega sin(\theta/2)cos(\theta/2) + i\dot{\beta} + \beta\omega sin^2(\theta/2) = \beta E_e$$

The third term i, as before, second order in ω, and we can get rid of it, as well as all higher terms in the expansion of the second half of the exponential in the first term:

$$e^{-itE_g}\omega sin(\theta/2)cos(\theta/2) + i\dot{\beta} = \beta E_e$$

Solve this equation by summing a the general solution to the homogeneous equation to the solution with the inhomogeneous term, and using the condition $\beta(0) = 0$, for the system starts in the ground state. The solution is:

$$\beta(t) = \frac{\omega sin\theta}{2(E_e - E_g)}(e^{-iE_g t} - e^{-iE_e t}) = \frac{i\omega sin\theta}{\gamma B_0} sin(\gamma B_0 t/2)$$

and the probability for excitation is:

$$P = |\beta(2\pi/\omega)|^2 = \frac{\omega^2 sin^2\theta}{\gamma^2 B_0^2} sin^2(\gamma B_0 \pi/\omega) \tag{3}$$