M01Q.1

1). The Hamiltonian is

$$H = -\mu \vec{B}\vec{\sigma} \tag{1}$$

$$= -\mu B_0 \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \tag{2}$$

The ground state is the spin state that directs parallel to the magnetic field. Solving the eigenstate of Hamiltonian, we get

$$|\theta, \phi\rangle_0 = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix} \tag{3}$$

2). In the adiabatic limit, at any time t, the wavefunction is still in the eigenstate of Hamiltonian $H(t)$, but up to a phase. The state can be written as

$$|\theta, \phi(t)\rangle = e^{-iE\_t/\hbar}\, e^{i\varphi(t)}\, |\theta, \phi(t)\rangle_0 \tag{4}$$

where $E_{-} = -\mu B_0$ is the ground state energy. $|\theta, \phi\rangle_0$ is the instantaneous eigenstate of $H(t)$.

Substitute $|\theta, \phi(t)\rangle$ into Schrodinger Equation

$$i\hbar \frac{\partial}{\partial t} |\theta, \phi\rangle = E_{-} |\theta, \phi\rangle \tag{5}$$

$$= E_{-} |\theta, \phi\rangle - \hbar \frac{\partial \varphi(t)}{\partial t} |\theta, \phi\rangle + i\hbar e^{-iE\_t/\hbar}\, e^{i\varphi(t)}\, \frac{\partial}{\partial t} |\theta, \phi\rangle_0 \tag{6}$$

, we get $\frac{\partial\varphi(t)}{\partial t} = \langle\theta,\phi|_0 i \frac{\partial}{\partial t} |\theta,\phi\rangle_0$. So, $\varphi(t) = \cos^2\frac{\theta}{2}\omega t$.

3). After a whole circle, the Hamiltonian ends up exactly the same as the initial Hamiltonian. Let's label $|+\rangle$ and $|-\rangle$ as the excited and ground state of $H(t=0)$, $|-\rangle = (\cos\theta/2, \sin\theta/2)^T$, $|+\rangle = (\sin\theta/2, -\cos\theta/2)^T$.

Split Hamiltonian to time independent part H_0 and time dependent part H' .
$H = -\mu\vec{B}\vec{\sigma} = H_0 + H'$, where $H_0 = -\mu B_0(\sigma_z\cos\theta + \sigma_x\sin\theta)$ and $H' = -\mu B_0(\sigma_x\sin\theta(\cos\phi - 1) + \sigma_y\sin\theta\sin\phi)$. Regard the time-dependent part H' as perturbation and use interaction picture to solve this question.

Use interaction picture, we get the transition amplitude, which is $A = \langle+|Texp\{\frac{-i}{\hbar}\int_t e^{iH_0t/\hbar}H'e^{-iH_0t/\hbar}dt\}|-\rangle$, where T is time ordering operator. Expand this formula to the lowest nonzero order of H' , we get

$$A = \frac{-i}{\hbar}\int_t dte^{i(E_+-E_-)t/\hbar}\langle+|H'|-\rangle \tag{7}$$
$$= \frac{\mu B_0 i}{\hbar}\int_t dte^{i2\mu B_0 t/\hbar}[-\cos^2\frac{\theta}{2}\sin\theta(e^{i\phi}-1) + \sin^2\frac{\theta}{2}\sin\theta(e^{-i\phi}-1)] \tag{8}$$

, where $\phi = \omega t$.

Finally, we get $A = \frac{\omega\sin\theta}{2}(e^{i2\pi c/\omega} - 1)[\frac{\cos^2\frac{\theta}{2}}{c+\omega} + \frac{\sin^2\frac{\theta}{2}}{c-\omega}]$, where $c = 2\mu B_0/\hbar$. The transition probability is $P = |A|^2$.

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2 thoughts on "M01Q.1"

M

December 17, 2013 at 2:17 am

Good. Now expand the solution.

The formula (1) is well-known and probably you don't have to explain it on the exam, but here it would make sense to give a short explanation/derivation.
In the second part of the problem, you probably substitute $|\theta, \phi(t)\rangle$ into the time-dependent Schroedinger equation, right? In the text you write stationary equation instead -- fix it.
Also, part 2 has to be expanded, I believe. You didn't write out explicitly the differential equation for $\varphi(t)$ which you solved to find the answer. The solution will benefit from adding some missing steps here.
Part 3 looks OK to me, although I did't check the computations.

J
December 17, 2013 at 11:53 am

Hi, Mykola, I have added more details in part 1 and part 2. Thank you.
