Problem M01Q.3

Under the 2D Born approximation, the scattering amplitude in a direction θ is given by

$$f_k(\theta) = -\frac{2m}{\hbar^2}\frac{1}{2\pi}\int \mathrm{d}^2x\; V(x)\, e^{iq\cdot x},$$

where q is the momentum transfer. Performing the Fourier transform gives

$$\begin{aligned}\int \mathrm{d}^2x\; V(x)\, e^{iq\cdot x} &= \lambda \sum_{n=-\infty}^{\infty}\int_{2na}^{(2n+1)a} e^{iq_x x}\,\mathrm{d}x \\ &= \frac{\lambda}{iq_x}\sum_{n=-\infty}^{\infty} e^{2niq_x a}(e^{iq_x a}-1) \\ &= \frac{\lambda}{iq_x}(e^{iq_x a}-1)\sum_{m\in\mathbb{Z}}\delta\Big(\frac{q_x a}{\pi}-m\Big).\end{aligned}$$

Assuming elastic scattering, we have $q_x = p\sin\theta$, so that the cross section for reflection (per unit length) is

$$\begin{aligned}\sigma_{\text{refl}} &= \frac{m^2}{\hbar^4\pi^2}\lambda^2\sum_{m\in\mathbb{Z}}\frac{\left|e^{im\pi}-1\right|^2}{(m\pi/a)^2} \\ &= \frac{4m^2a^2\lambda^2}{\hbar^4\pi^4}\sum_{m\text{ odd}}\frac{1}{m^2} \\ &= \frac{4m^2a^2\lambda^2}{\hbar^4\pi^4}\left[\sum_{m\in\mathbb{Z}}\frac{1}{m^2}-\sum_{m\text{ even}}\frac{1}{m^2}\right] \\ &= \frac{4m^2a^2\lambda^2}{\hbar^4\pi^4}\left[\frac{\pi^2}{6}-\frac{1}{4}\sum_{m\in\mathbb{Z}}\frac{1}{m^2}\right] \\ &= \frac{a^2}{2\pi^2}\left(\frac{m\lambda}{\hbar^2}\right)^2,\end{aligned}$$

where we have regulated the delta functions since the scattered distribution has finite support. (This occurs since the scattering potential has infinite extent.)

As expected for a 1D cross section per unit length, the resulting expression for σ_{refl} is dimensionless. (Recall that λ has units of [energy]·[length].)

Hence the probability of transmission is

$$P_{\text{trans}} = \boxed{1-\frac{a^2}{2\pi^2}\left(\frac{m\lambda}{\hbar^2}\right)^2}.$$