M01M.3

Solution to M01M.3 - Particle in Gravitational and Magnetic Fields

1. Using Newton's 2nd law, we can write two equations for $\dot{v_x} = a_x$ and $\dot{v_y} = a_y$. The particle feels a Lorentz force $F_{mag} = \frac{q}{c}(\vec{v} \times \vec{B})$ and a gravitational force $F_g = mg\hat{j}$. Our two equations are then:

$$\dot{v_x} = \frac{Bq}{mc} v_y$$

$$\dot{v_y} = g - \frac{Bq}{mc} v_x$$

We have these coupled equations as a result of the cross product in the Lorentz force. We make the substitution $\omega = \frac{Bq}{mc}$ in anticipation of the following step. Taking the time derivative of the $\ddot{v_y}$ equation, we find:

$$\ddot{v_y} = -\omega\dot{v_x} = -\omega^2 v_y$$

The solution to this equation is $v_y = A\cos(\omega t) + B\sin(\omega t)$. We know from the initial condition $v_y(t=0) = 0$ that $A = 0$. Plugging this into our equations for $\dot{v_x}$:

$$\dot{v_x} = \omega v_y$$
$$\Rightarrow \ v_x = -B\cos(\omega t) + D$$
$$\Rightarrow \ v_x(t=0) = 0 \ \Rightarrow \ D = B$$

To finish finding v_x and v_y we substitute our solutions into our earlier equation for the force in the $\hat{j}$ direction:

$$\dot{v_y} + v_x = g$$
$$\Rightarrow \ \omega B\cos(\omega t) - \omega B\cos(\omega t) + D\omega = g$$

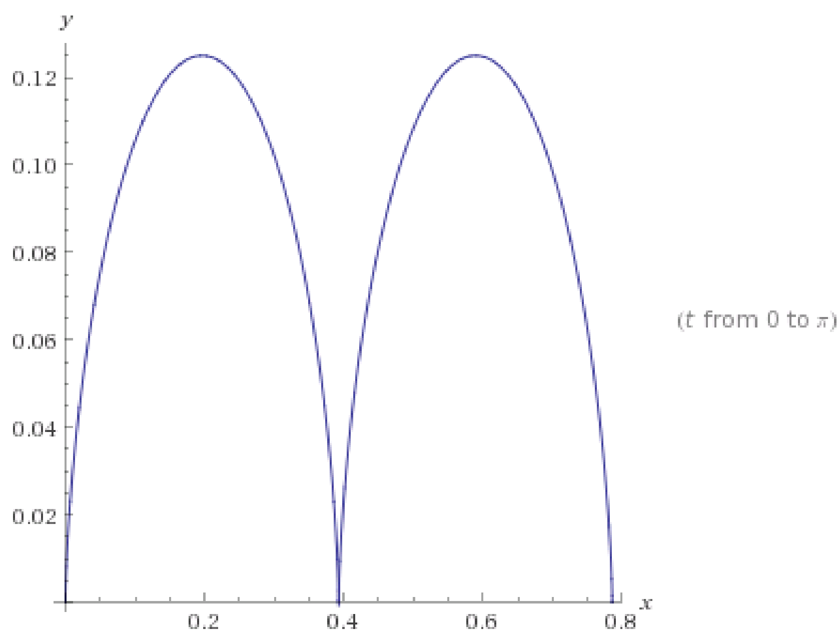
$$\Rightarrow \; D = \frac{g}{\omega} = B$$

We integrate again to find $x(t), y(t)$:

$$x(t) = \frac{g}{\omega}t - \frac{g}{\omega^2}\sin(\omega t) + E$$
$$y(t) = -\frac{g}{\omega^2}\cos(\omega t) + F$$

And using the initial conditions $x(t=0) = 0, \; y(t=0) = 0$: $E = 0, \; F = \frac{g}{\omega^2}$.

Plotted on WolframAlpha for $g = 1, \; \omega = 4, \; 0 < t < \pi$:



2. We add our drag force $-\beta v$ to our earlier equations. To compute the terminal velocity, we take $\dot{\vec{v}} = 0$:

$$mg\hat{j} + \frac{qB}{c}(v_y\hat{i} - v_x\hat{j}) - \beta(v_x\hat{i} + v_y\hat{j}) = 0$$

Taking $\omega = \frac{qB}{mc}$ and $\Omega = \frac{\beta}{m}$, we arrive at two equations, one for the $\hat{i}$ and $\hat{j}$ directions respectively:

$$\omega v_y^{term} - \Omega v_x^{term} = 0$$
$$g - \omega v_x^{term} - \Omega v_y^{term} = 0$$

These lead us to the following results, after solving for v_x^{term} and substituting:

$$v_y^{term} = \frac{g\Omega}{\omega^2+\Omega^2}$$
$$v_x^{term} = \frac{g\omega}{\omega^2+\Omega^2}$$

$$\Rightarrow v^{term} = \sqrt{v_x^2 + v_y^2} = \frac{g}{\sqrt{\omega^2 + \Omega^2}}$$

For non-zero a_x, a_y we can treat the solutions to the above equations of motion as a system of linear first-order equations for $\dot{v_x}, \dot{v_y}$. Taking one time derivative of the equations yields:

$$\ddot{v_x} = \omega\dot{v_y} - \Omega\dot{v_x}$$
$$\ddot{v_y} = -\omega\dot{v_x} - \Omega\dot{v_y}$$

Now taking $\dot{v_x} = Ae^{mt}, \dot{v_y} = Be^{mt}$, we derive the following equations which must be satsfied by A, B, and m:

$$(m + \Omega)A - \omega B = 0$$
$$\omega A + (m + \Omega)B = 0$$

Nontrivial solutions occur when the determinant of the matrix of coefficients for $A, B = 0$. This produces the following quadratic, which we can quickly solve:

$$(m + \Omega)^2 - (-\omega^2) = 0$$
$$\Rightarrow m = -\Omega \pm i\omega$$

Taking $m_\pm = -\Omega \pm i\omega$, we put these values back into our equations and solve for conditions on A, B. We find:

For $m_+ : iA = B$ For $m_- : -iA = B$

Thus our solutions take the form:

$$\dot{v_x} = Ae^{m_+t} + Be^{m_-t}$$
$$\dot{v_y} = iAe^{m_+t} - iBe^{m_-t}$$

Integrating:

$$v_x = \frac{A}{m_+}e^{m_+t} + \frac{B}{m_-}e^{m_-t} + C$$
$$v_y = \frac{iA}{m_+}e^{m_+t} - \frac{iB}{m_-}e^{m_-t} + D$$

Our initial conditions require C, D cancel the contributions from the exponentials for $t = 0$. After much strife, we can also find the values of A and B given our original equations of motion: $B = -A,\ A = \frac{-ig}{2}$. We further find that $C = v_x^{term}, D = v_y^{term}$. Then integrating once more, we find:

$$x(t) = \frac{ig}{2}\left(\frac{e^{m_- t}}{m_-^2} - \frac{e^{m_+ t}}{m_+^2}\right) + v_x^{term} t + E$$
$$y(t) = \frac{g}{2}\left(\frac{e^{m_+ t}}{m_+^2} + \frac{e^{m_- t}}{m_-^2}\right) + v_y^{term} t + F$$

where taking the real part is implied and $E,\ F$ enforce the initial conditions $x(t=0)=0, y(t=0)=0$. Thus $E = -\frac{ig}{2}\left(\frac{1}{m_-^2} - \frac{1}{m_+^2}\right)$, and $F = -\frac{g}{2}\left(\frac{1}{m_+^2} + \frac{1}{m_-^2}\right)$.

3. The force applied to the particle as a result of its radiation is $\propto \ddot{\vec{v}}$. We can consider this radiation as reducing the radial motion of the particle around the "center" of its loop, this center drifting along the $\hat{i}$ direction. At long times, the radial motion of the particle about this moving center will be 0, as we can see from imagining a "terminal velocity" in our original set of equations in part 1. Eventually the particle moves in the $\hat{i}$ direction with velocity $v^{term} = \frac{g}{\omega}$, it being assumed that the velocity at this time is constant which means the particle does not feel the radiation force.

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2 thoughts on "M01M.3"

K

October 11, 2013 at 4:23 pm

Thanks M. I've added the appropriate constants to my solution for part 2) and included a plot that wasn't created with a typo.

M

October 10, 2013 at 5:42 pm

Good. So...

1) Your solution of part 1 is correct, but the plot for your (x(t), y(t)) looks different from what you presented. Try to build the plot one more time. Notice that the curve should be cycloid as follows from the

parametrization.

2) Generally, you're doing it right. But the answer doesn't seem to satisfy the initial condition $x(0) = y(0) = 0$. Try to find the correct answer and plot it please.
Also it could be very helpful to consider variable $\zeta = x + iy$ and derive a differential equation on it. In such a case you would have to solve only one equation, which would serve both the part 1 and part 2. However, you don't have to and can do it your way.

3) Okay.