Solution to M01M.3 — Particle in Gravitional and Magnetic Fields

1.

Without loss of generality, we assume the particle has positive charge q. The equation of motion for the particle in gravitational and magnetic fields is:

$$q\vec{v}\times\vec{B}+m\vec{g}=m\dot{\vec{v}} \tag{1}$$

One of the ideas to solve this problem is to split particle's motion into several independent, simpler motions.

Actually, we can view the particle's motions as the combinition of three independent motions, $\vec{v}=\vec{v}_{||}+\vec{v}_u+\vec{v}_c$. $\vec{v}_{||}$ is along the magnetic field, which is only accelerated by $\vec{g}_{||}$, since there is no lorentz force in the direction parrel to $\vec{B}$. Both $\vec{v}_u$ and $\vec{v}_c$ are perpendicular to $\vec{B}$. $\vec{v}_u$ is the uniform linear motion, whose lorentz force cancels the gravitational force $m\vec{g}_\perp$, and thus the net force is zero for it. $\vec{v}_c$ is the cyclotron motion.
Split the gradational field as $\vec{g}=\vec{g}_{||}+\vec{g}_\perp$, where $\vec{g}_{||} \mid\mid \vec{B}$, $\vec{g}_\perp\perp\vec{B}$.

We write down the equation of motions as

$$m\vec{g}_{||}=m\dot{\vec{v}}_{||} \tag{2}$$

$$q\vec{v}_u\times\vec{B}+m\vec{g}_\perp=0 \tag{3}$$

$$q\vec{v}_c\times\vec{B}=m\dot{\vec{v}}_c \tag{4}$$

Choose a coordinate such that $\hat{z}=\hat{k}$, $\hat{y}=(\hat{j}-\cos\theta\hat{k})/\sin\theta$, $\hat{x}=\hat{y}\times\hat{z}$. We can easily get that $\vec{g}_{||}=g\cos\theta\hat{z}$, $\vec{g}_\perp=g\sin\theta\hat{y}$, where $\cos\theta=\hat{j}\cdot\hat{k}$.

$$\dot{v_{||}} = g_{||}t\hat{z} = g\cos\theta t\hat{z} \tag{5}$$

$$\vec{v}_u = \frac{g_\perp}{\omega}\hat{x} = \frac{g\sin\theta}{\omega}\hat{x} \tag{6}$$

$$\vec{v}_c = \frac{g_\perp}{\omega}[-\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}] = \frac{g\sin\theta}{\omega}[-\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}] \tag{7}$$

where $\hat{j}\cdot\hat{k} = \cos\theta, \omega = \frac{qB}{m}$.

So,

$$\vec{v} = \frac{g\sin\theta}{\omega}[1-\cos(\omega t)]\hat{x} + \frac{g\sin\theta}{\omega}\sin(\omega t)\hat{y} + g\cos\theta t\hat{z} \tag{8}$$

We get

$$\vec{r} = \frac{g\sin\theta}{\omega^2}[\omega t - \sin(\omega t)]\hat{x} + \frac{g\sin\theta}{\omega^2}[1-\cos(\omega t)]\hat{y} + \frac{1}{2}g\cos\theta t^2\hat{z} \tag{9}$$

Usually, the particle's motion is not confined in a plane. Except $\vec{g} \,||\, \vec{B}$, which is a trivial case, or $\vec{g} \perp \vec{B}$, where $\hat{j}\cdot\hat{k} = 0$. The motion becomes just $\vec{r} = \frac{g}{\omega^2}[\omega t - \sin(\omega t)]\hat{x} + \frac{g}{\omega^2}[1-\cos(\omega t)]\hat{y}$.

2.

The equation of motion in the presence of drag force is

$$q\vec{v}\times\vec{B} + m\vec{g} - \beta\vec{v} = m\dot{\vec{v}} \tag{10}$$

Similarly, split $\vec{v}$ as $\vec{v} = \vec{v}_{||} + \vec{v}_u + \vec{v}_c$, where they satisfy

$$m\vec{g}_{||} - \beta\vec{v}_{||} = m\dot{\vec{v}}_{||} \tag{11}$$

$$q\vec{v}_u\times\vec{B} + m\vec{g}_\perp - \beta\vec{v}_u = 0 \tag{12}$$

$$q\vec{v}_c\times\vec{B} - \beta\vec{v}_c = m\dot{\vec{v}}_c \tag{13}$$

Also choose $\hat{z} = \hat{k}$, $\hat{y} = (\hat{j} - \cos\theta\hat{k})/\sin\theta$, $\hat{x} = \hat{y}\times\hat{z}$. We can get

$$\vec{v}_{||} = \frac{mg\cos\theta}{\beta}(1-e^{-\beta t/m})\hat{z} \tag{14}$$

$$\vec{v}_u = \frac{\omega g\sin\theta}{\omega^2 + (\beta/m)^2}[\hat{x} + \frac{\beta}{m\omega}\hat{y}] \tag{15}$$

$$\vec{v}_c = \frac{\omega g\sin\theta}{\omega^2 + (\beta/m)^2}e^{-\beta t/m}[(-\cos\omega t - \frac{\beta}{m\omega}\sin\omega t)\hat{x} + (\frac{-\beta}{m\omega}\cos\omega t + \sin\omega t)\hat{y}] \tag{16}$$

So, the particle's motion is

$$\vec{v} = \frac{\omega g \sin\theta}{\omega^2 + (\beta/m)^2}(1 - e^{-\beta t/m}\cos\omega t - \frac{\beta}{m\omega}e^{-\beta t/m}\sin\omega t)\hat{x} \\ + \frac{\omega g \sin\theta}{\omega^2 + (\beta/m)^2}(\frac{\beta}{m\omega} - e^{-\beta t/m}\cos\omega t + e^{-\beta t/m}\sin\omega t)\hat{y} \\ + \frac{mg\cos\theta}{\beta}(1 - e^{-\beta t/m})\hat{z} \tag{17}$$

The final velocity is

$$\vec{v}_f = \frac{\omega g \sin\theta}{\omega^2 + (\beta/m)^2}[\hat{x} + \frac{\beta}{m\omega}\hat{y}] + \frac{mg\cos\theta}{\beta}\hat{z} \tag{18}$$

3.

The uniform linear motion $\vec{v}_u$ will not have electromagnetic radiation. While both $\vec{v}_c$ and $\vec{v}_{||}$ will have electromagnetic radiation. The power of electromagnetic radiation is proportional to $|\vec{a}|^2 = g^2$, where $\vec{a}$ is the acceleration, according to Larmor Formula. The final velocity is still $\vec{u} = \frac{g\sin\theta}{\omega}\hat{x}$.

One thought on "Solution to M01M.3 — Particle in Gravitional and Magnetic Fields"

December 3, 2013 at 4:44 am

Your answers make sense, but I don't really like the approach in which you initially split the motion into several types -- you somehow assume that you know that the particle is going to have a uniform motion component.