M01M.3—Particle in Gravitaitonal and Magnetic Fields

Problem

A particle of mass m and charge q moves freely in a gravitational field $\mathbf{g} = g\hat{j}$ and a magnetic field $\mathbf{B} = B\hat{k}$. At time $t = 0$ the particle is released from the origin O with no initial velocity. It traces a curve in the x - y plane.

a) Find the parametric equations $x = x(t),\ y = y(t)$ describing the curve. Sketch the curve on an x - y diagram.

The above motion is idealized, because two effects have been ignored: air drag and radiation damping.

b) Now assume that the particle also feels a drag force due to the surrounding atmosphere, $F = -\beta v$. Derive the motion of the particle. What is its final velocity?

c) Instead of air drag, suppose we include the damping effect caused by the electromagnetic radiation emitted during its motion. Describe, qualitatively, how this modifies the motion found in part a). What is the final velocity of the particle?

Solution

a)

The particle is subject to only two forces, the gravitational force and the Lorentz force. It's motion can be described in Cartesian coordinates where $\mathbf{x} = (x(t), y(t), z(t))$ and we have the initial conditions $\mathbf{x}(0) = \dot{\mathbf{x}}(0) = 0$. Using Newton's second law:

$$m\ddot{\mathbf{x}} = m\mathbf{g} + q(\dot{\mathbf{x}} \times \mathbf{B})$$
$$m\ddot{\mathbf{x}} = mg\hat{j} + q(\dot{\mathbf{x}} \times B\hat{k})$$
$$m\ddot{\mathbf{x}} = mg\hat{j} + q(\dot{y}B\hat{i} - \dot{x}B\hat{j})$$

Giving us the system of equations:
(1) $m\ddot{x} = qB\dot{y}$
(2) $m\ddot{y} = mg - qB\dot{x}$
(3) $m\ddot{z} = 0$

Since $z(0) = \dot{z}(0) = 0$, (3) implies $z(t) = 0$, so we can ignore the z-component for the rest of the problem and concern ourselves only with the x and y components.

Taking the derivative of (2) with respect to time gives $\ddot{x} = -\frac{m}{qB}\dddot{y}$. Subbing this into (1) we get $\dddot{y} = -(\frac{qB}{m})^2\dot{y}$

This is a simple harmonic oscillator equation. Letting $\omega = \frac{qB}{m}$, the general solution is $\dot{y}(t) = C_1\cos\omega t + C_2\sin\omega t$

The initial condition $\dot{y}(0) = 0$ implies $C_1 = 0$. Since the only force acting on the particle at $t = 0$ is gravity, we also have the condition that $\ddot{y}(0) = g$. This implies that $C_2 = \frac{g}{\omega}$. So far we have $\dot{y}(t) = \frac{g}{\omega}\sin\omega t$

Integrating, and using the condition $y(0) = 0$,
$y(t) = \frac{g}{\omega^2}(1 - \cos\omega t)$

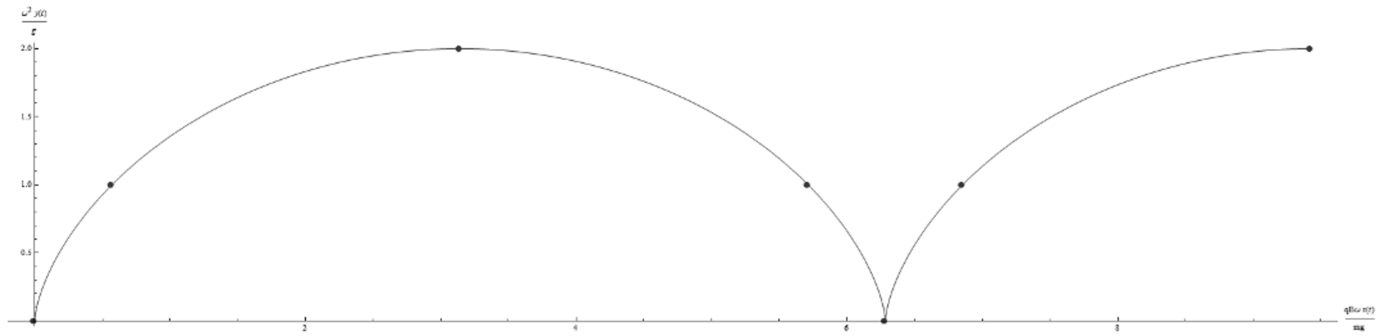
Now subbing $\ddot{y} = g\cos\omega t$ into (2) gives
$\dot{x}(t) = \frac{g}{\omega}(1 - \cos\omega t)$
$x(t) = \frac{g}{\omega^2}(\omega t - \sin\omega t + C_3)$

The initial condition $x(0) = 0$ implies $C_3 = 0$ and we have our solution:

$$\boxed{\begin{aligned} x(t) &= \frac{g}{\omega^2}(\omega t - \sin\omega t) \\ y(t) &= \frac{g}{\omega^2}(1 - \cos\omega t) \end{aligned}}$$

Taking simple values of time can give us a clue to what the curve looks like.

$Time$	$\frac{qB\omega}{mg}x(t)$	$\frac{\omega^2}{g}y(t)$
0	0	0
$\frac{\pi}{2\omega}$	$\frac{\pi}{2} - 1$	1
$\frac{\pi}{\omega}$	π	2
$\frac{3\pi}{2\omega}$	$\frac{3\pi}{2} + 1$	1
$\frac{2\pi}{\omega}$	2π	0



b)

The drag force is proportional to the velocity in each direction, so the new equations of motion are
(4) $m\ddot{x} = qB\dot{y} - \beta\dot{x}$
(5) $m\ddot{y} = mg - qB\dot{x} - \beta\dot{y}$

To solve this system we introduce a new coordinate $\xi = x + iy$. This reduces the system to one differential equation:

$$m\ddot{\xi} = -(iqB + \beta)\dot{\xi} + img$$

This is the same form as part (a) would take with the same method, with a different factor of $\dot{\xi}$. Part (a) had solutions that were a mix of sinusoidal, linear, and constant terms. So for part (b) we use an ansatz of the form:
(6) $\xi(t) = C_1 e^{i\Omega t} + C_2 t + C_3$

Where C_1, C_2, C_3, Ω can all be complex. Subbing this in we get

$$-mC_1\Omega^2 e^{i\Omega t} = -(iqB + \beta)(i\Omega C_1 e^{i\Omega t} + C_2) + img$$

Equate constant terms and coefficients of $e^{i\Omega t}$

$$\Omega = \frac{i\beta - qB}{m} = \frac{\beta^2 + q^2B^2}{-qBm - im\beta}$$
$$C_2 = \frac{img}{iqB + \beta} = \frac{mgqB + imgβ}{q^2B^2 + \beta^2} = -g\Omega^{-1}$$

Using initial conditions $\dot{\xi}(0) = 0$ gives $C_1 = \frac{iC_2}{\Omega} = -ig\Omega^{-2}$. Then $\xi(0) = 0$ gives $C_3 = -C_1 = ig\Omega^{-2}$. Plugging all these back into the ansatz (6):

$$\xi(t) = -ig\Omega^{-2}e^{i\Omega t} - g\Omega^{-1}t + ig\Omega^{-2}$$
$$\xi(t) = -im^2 g\frac{q^2B^2 - \beta^2 + 2iqB\beta}{(q^2B^2 + \beta^2)^2} e^{\frac{-\beta}{m}t} e^{\frac{-iqB}{m}t} + \frac{qB + i\beta}{q^2B^2 + \beta^2} mgt + im^2 g\frac{q^2B^2 - \beta^2 + 2iqB\beta}{(q^2B^2 + \beta^2)^2}$$

We now recover $x(t) = Re(\xi(t))$and $y(t) = Im(\xi(t))$:

$$x(t) = m^2 g \frac{2qB\beta \cos \frac{qB}{m} t - (q^2 B^2 - \beta^2) \sin \frac{qB}{m} t}{(q^2 B^2 + \beta^2)^2} e^{\frac{-\beta}{m} t} + \frac{qB}{(q^2 B^2 + \beta^2)^2} mgt - m^2 g \frac{2qB\beta}{(q^2 B^2 + \beta^2)^2}$$

$$y(t) = m^2 g \frac{(\beta^2 - q^2 B^2) \cos \frac{qB}{m} t - 2qB\beta \sin \frac{qB}{m} t}{(q^2 B^2 + \beta^2)^2} e^{\frac{-\beta}{m} t} + \frac{\beta}{(q^2 B^2 + \beta^2)^2} mgt + m^2 g \frac{q^2 B^2 - \beta^2}{(q^2 B^2 + \beta^2)^2}$$

Now to find the terminal velocity. Due to the drag, the sinusoidal motion in part (a) will decay until the particle reaches terminal velocity and the gravitational force is equal and opposite to the Lorentz force, at which point the particle continues in a straight line at a constant speed. In other words, when the particle reaches its final velocity, $\ddot{x} = \ddot{y} = 0$. Then equations (4) and (5) give:

(7) $0 = qB\dot{y} - \beta\dot{x}$

(8) $0 = mg - qB\dot{x} - \beta\dot{y}$

Eliminating $\dot{x}$,

$$\dot{y} = \frac{mg}{\frac{q^2 B^2}{\beta} + \beta}$$

Subbing back into (6),

$$\dot{x} = \frac{qB}{\beta} \left(\frac{mg}{\frac{q^2 B^2}{\beta} + \beta} \right)$$

$$v_{final}^2 = \dot{x}^2 + \dot{y}^2 = \frac{q^2 B^2}{\beta^2} \dot{y}^2 + \dot{y}^2 = (1 + \frac{q^2 B^2}{\beta^2}) \left(\frac{mg}{\frac{q^2 B^2}{\beta} + \beta} \right)^2 = \frac{m^2 g^2}{q^2 B^2 + \beta^2}$$

$$v_{final} = \frac{mg}{\sqrt{q^2 B^2 + \beta^2}}$$

c)

Now we introduce damping radiation instead of drag. Damping radiation is proportional to $\frac{d^2 \vec{v}}{dt^2}$. The equations of motion are then:

(9) $m\ddot{x} = qB\dot{y} - \gamma\dddot{x}$

(10) $m\ddot{y} = mg - qB\dot{x} - \gamma\dddot{y}$

At equilibrium $\ddot{x} = \ddot{y} = \dddot{x} = \dddot{y}$

(11) $0 = qB\dot{y}$

(12) $0 = mg - qB\dot{x}$

$$v_{final}^2 = \dot{x}^2 + \dot{y}^2 = \frac{mg^2}{qB} + 0$$

$$v_{final} = \frac{mg}{qB}$$

So in this case the cycloid motion becomes damped until the particle is travelling parallel to the x-axis with enough speed for the Lorentz force to cancel out the gravitational force.

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3 thoughts on “M01M.3—Particle in Gravitaitonal and Magnetic Fields”

M
September 29, 2013 at 6:36 pm

Are you still working on part (c)?
Recall that that there is a damping force due to the radiation, which for the single particle is given by $\frac{2q^2}{3c^3}\frac{d^2\vec{v}}{dt^2}$ (the exact proportionality coefficient depends on units).

C
October 2, 2013 at 11:42 am

Thank you, I was unsure how to model the damping force due to radiation.

M
September 29, 2013 at 6:27 pm

The part (a) is almost perfect, except that in the final answer you are using both ω and $\frac{qB}{m}$ which are the same, and so it would be nicer to have either only ω or $\frac{qB}{m}$. In that case you will clearly see that the answer is just a cycloid (a curve drawn by a point on the rim of the rolling wheel), which is kind of cool.

Part (b) of the problem actually asks you to find an explicit form of the curve, I assume. You can find it with the same method you used in (a). However, there exists a nice trick helping to solve your equations (1) and (2) without extra differentiation, and which can be straightforwardly generalized to part (b) as well. Consider a new variable:

$$\xi = x + iy,$$

and take the linear combination of the equations (1) + i(2). As a result you'll get a single complex differential equation for $\xi(t)$, which will be completely equivalent to the system (1)-(2). This equation is quite simple.

If you apply the same technique to (b), you'll see that it works perfectly well, and that you won't have to solve any new differential equation at all.
