M01M.2

Solution to M01M.2 — Particle in an Anharmonic Potential

a) A plot of such potential is shown below. The particle was initially swinging high high up on the potential well as the result of large initial velocity. As the small damping gradually takes away energy, the amplitude decreases each oscillation while the period increases until it reaches the point where the total energy is 0 (note, it is possible to have negative total energy, which simply corresponds to particle swing in one of the slots). The period increases because the particle is spending more and more time passing through the center part of (x=0) potential well.

b) Following the suggestion of hint, let's write down the expression for period in a integral form. Suppose there isn't any friction. Conservation of energy implies

$$E = \frac{1}{2}m\dot{x}^2 - ax^2 + bx^4 \tag{1}$$

Rearrange eqn 1 to get an expression for dt/dx in terms of x. Symmetry of potential indicates that one full period is twice of the time it takes from x_- to x_+. Therefore,

$$T = 2\int_{x_-}^{x_+} \frac{dx}{\sqrt{\frac{2}{m}\left(E + ax^2 - bx^4\right)}} \tag{2}$$

Of course, the oscillation amplitude is related to total energy, as we can find it by using conservation of energy at x_{+-}

$$-ax^2 + bx^4 = E \tag{3}$$

which gives

$$x^2 = \frac{a + \sqrt{a^2 + 4bE}}{2b} \approx \frac{a}{b} + \frac{E}{a} \tag{4}$$

Where I demand x^2 to be positive in finding the root. Continue expanding to the linear order, we eventually have $\|x_{+-}\| = \sqrt{\frac{a}{b}}\left(1 + \frac{bE}{2a^2}\right)$.

Now in the presence of small damping, the energy in above expression is no longer a constant. As I mentioned before, the particle will spend major of their time passing through the central region of potential when E is small. Hence we can neglect the x^4 terms since it does not contribute much to the integral.

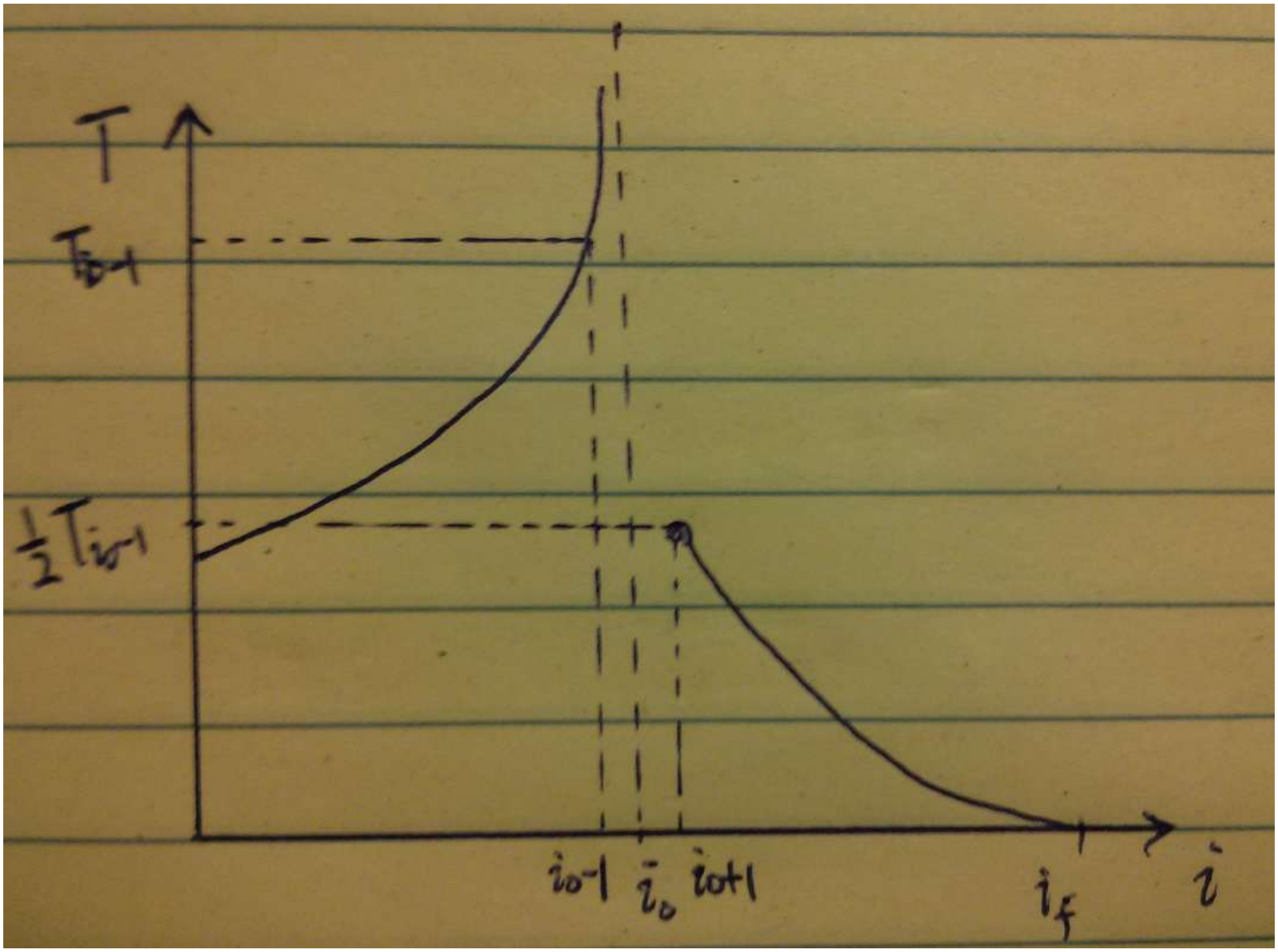
$$\begin{aligned} T = 2\int_{x_-}^{x_+} \frac{dx}{\sqrt{\frac{2}{m}(E + ax^2)}} &= 2\sqrt{\frac{2m}{a}} \int_0^{x_+} \frac{dx}{\sqrt{E/a + x^2}} \\ &= 2\sqrt{\frac{2m}{a}} \ln\left[\frac{x_+ + \sqrt{E/a + x_+^2}}{\sqrt{E/a}}\right] \end{aligned} \tag{5}$$

In the region of small positive E, we expand the square root term and obtain

$$T \sim \log(2x_+/\sqrt{E/a}) = \log\left(2\sqrt{\frac{a}{b} + \frac{E}{a}}/\sqrt{E/a}\right) = \log\left(2\sqrt{\frac{a^2}{bE} + 1}\right) \tag{6}$$

Since $1/E$ dominates, omit 1 and $T \sim \log E$. Suppose the damping is small so that the energy decreases linearly with i, then $T \sim \log(i - i_0)$ where $i \rightarrow i_0^+$.

c) The period peaks at i_0 (actually goes to infinity as you can see from the expression for T) and then suddenly decrease afterward and eventually hit 0 when damping exhausts all the energy in the system at a very large i_f (it would make more physical sense in classical mechanics). But what T immediately after i_0? Looking at the plot of potential, we realize the period after i_0 is actually half of the period right before i_0, or $T_{i_0+1} = T_{i_0-1}/2$. Here we go,



2 thoughts on “M01M.2”

December 16, 2013 at 1:44 am

Okay, now better.
You may notice that you actually didn't need the expression for x_+, as the main contribution came from x close to zero.

December 8, 2013 at 8:34 pm

Your solution goes in a correct direction, but you should be more accurate in making an estimate for T when E is close to zero. And actually, I don't see your answer for the dependence of T on i.