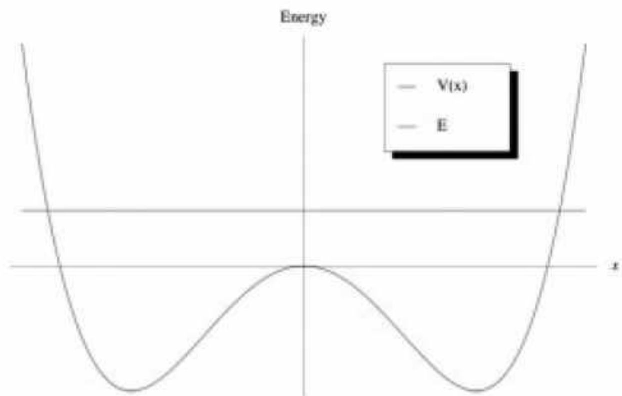
M01M.2



A plot of the potential energy as a function of x is shown above together with a horizontal line indicating the total energy of the particle. Under light damping, we can assume that the total energy of the particle is approximately constant during one whole oscillation and the plot above can be intepreted to show the approximate, constant energy of the particle during the ith oscillation, with the turning points x_+ and x_- located at where E intersects $V(x)$.

Since the partice was launched with a large initial velocity, $E_{initial}$ is above 0. However, due to the light damping, E will decreases slowly. As it approaches $E = 0$, the period will increase quite significantly near $E = 0$. This is due to the fact that at $x = 0$ and $E = 0$, the total energy equals the potential energy and hence the kinetic energy, together with the particle's velocity, vanishes, which results in T increasing sharply.

Assuming that the energy is approximately constant in a single oscillation under light damping, we can write that

$$\frac{T}{2} = \int dt = \int \frac{dt}{dx} dx = \int_{x_-}^{x_+} \frac{1}{\dot{x}} dx \tag{1}$$

We have from energy conservation

$$\begin{aligned} E &= \frac{1}{2}m\dot{x}^2 - ax^2 + bx^4 \\ \implies \dot{x} &= \sqrt{\frac{2}{m}}\sqrt{E + ax^2 - bx^4} \end{aligned} \tag{2}$$

Substituting into (1), we obtain:

$$T_i = 2\sqrt{\frac{m}{2}}\int_{x_-}^{x_+} \frac{1}{\sqrt{E_i + ax^2 - bx^4}}dx \tag{3}$$

where the subscript i indicates that (3) is valid for the ith oscillation where $E = E_i$. As $E \to 0$, the integrand in (3) approaches infinity as $x \to 0$ and as $x \to x_\pm$ since by definition of $x_\pm$, $E_i - \left(-ax_\pm^2 + bx_\pm^4\right) = 0$. It is therefore helpful to re-write (3) as:

$$T_i = 4\sqrt{\frac{m}{2}}\left(\int_\epsilon^0 f(x)dx + \int_{x_+-\epsilon}^{\epsilon} f(x)dx + \int_{x_+}^{x_+-\epsilon} f(x)dx\right) \tag{4}$$

where $f(x)$ is the integrand in (3) and where we have made use of the fact that $f(x)$ is even and $x_+ = -x_-$ because $V(x)$ is even. In order to determine which integral in (4) dominates as $E \to 0$, we note that in the case of light damping,

$$\begin{aligned} F \approx -\frac{dV}{dx} &= -2ax + 4bx^3 = m\ddot{x} \\ \implies \ddot{x} &= \frac{1}{m}\left(-2ax + 4bx^3\right) \end{aligned} \tag{5}$$

Since the integrand approaches infinity as $E \to 0$ only as $x \to 0$ and $x \to x_\pm$, it is sufficient to examine the first and third integral in (4) to determine which integral dominates (4).

$$T = \int \frac{1}{\dot{x}}dx = \int \frac{1}{\int \ddot{x}dt}dx = \int \frac{1}{\int \ddot{x}dx\,\frac{dt}{dx}}dx = \int \frac{1}{\int \frac{\ddot{x}}{\dot{x}}\,dx}dx' \tag{6}$$

Whence,

$$T_{\text{1st integral}} = \int_\epsilon^0 \frac{1}{\int_{x_+}^{x'} \frac{\ddot{x}}{\dot{x}}\,dx}dx' \tag{7}$$

$$T_{\text{3rd integral}} = \int_{x_+}^{x_+-\epsilon} \frac{1}{\int_{x_+}^{x'} \frac{\ddot{x}}{\dot{x}}\,dx}dx' \tag{8}$$

From (5), it is evident that for sufficiently small ϵ, $\ddot{x} \approx 0$ for $0 \leq x \leq \epsilon$ but $\ddot{x}$ is close to its maximum for $x_+ - \epsilon \leq x \leq x_+$.

Therefore, the denominator in (7) is smaller than the denominator in (8) even as $\dot{x} \to 0$ in both cases. Consequently, (7) will be dominant compared to (8) and we have from (4):

$$T_i \approx 4\sqrt{\frac{m}{2}} \int_{\epsilon}^{0} f(x)dx \tag{9}$$

In order to evaluate (9), we first expand $f(x)$ around $x = 0$:

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots \\ &\approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\ &= \frac{1}{\sqrt{E_i}} - \left(\frac{1}{\sqrt{E_i}}\right)^3 \frac{a}{2}x^2 \end{aligned} \tag{10}$$

Substituting (10) into (9) and performing the integral, we get:

$$\begin{aligned} T_i &\approx 4\sqrt{\frac{m}{2}}\left(-\frac{\epsilon}{\sqrt{E_i}} + \frac{a}{6}\left(\frac{\epsilon}{\sqrt{E_i}}\right)^3\right) \\ \implies T_i &\sim \left(\frac{1}{\sqrt{E_i - E_{i_0}}}\right)^3 \end{aligned} \tag{11}$$

where $E_{i_0} = 0$ is the energy of the i_0 th oscillation.

Moreover, under the assumptions of light damping where the energy in a single oscillation is approximately constant, the energy lost in one oscillation is given approximately by

$$E_i - E_{i+1} = 2\int \gamma \dot{x} dx = 2\int_{x_-}^{x_+} \sqrt{\frac{2}{m}}\sqrt{E_i + ax^2 - bx^4}\,\gamma dx \tag{12}$$

where γ here is the damping constant. But for i close to i_0, $E_i \approx 0$, which implies that

$$E_i - E_{i+1} \approx 2\int \gamma \dot{x} dx = 2\int_{x_-}^{x_+} \sqrt{\frac{2}{m}}\sqrt{ax^2 - bx^4}\,\gamma dx \equiv A \tag{13}$$

where A is some constant. Hence, for i close to i_0,

$$\begin{aligned} E_i &= E_{i_0} + (i - i_0)A \\ \implies T_i &\sim \left(\frac{1}{\sqrt{i - i_0}}\right)^3 \end{aligned} \tag{14}$$

for light damping, i close to i_0 and $i > i_0$.

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2 thoughts on “M01M.2”

October 10, 2013 at 7:09 pm

You've sufficiently improved your solution.
However notice that your expansion (10) doesn't look good as E_i is small, and so $\frac{1}{\sqrt{E_i}}$ is big, and so the second non-zero term in your expansion is not expected to be smaller than the first one.
If you want to make some expansion which is definitely legitimate, you should find a small dimensionless parameter first.
In this case it will change your answer again.

October 8, 2013 at 4:56 pm

Approximation between (4) and (5) is not legitimate. We are interested in the regime when E_i comes close to zero. In such case indeed the main contribution comes from integration between $-\epsilon$ and ϵ. But you can't approximate it by rectangular -- you actually have to compute the leading order of the integral. Also, you should do some estimates to prove that indeed that integral gives the main contribution.

P.S. the answer will be different.