M01M.2

Solution to M01M.2 — Particle in an Anharmonic Potential

Firstly, we investigate the features of the potential. Because equation

$$V(x) = 0$$

has three roots

$$x_1 = 0, x_2 = \sqrt{a/b}, x_3 = -\sqrt{a/b}$$

and equation

$$\frac{\partial V}{\partial x} = 0$$

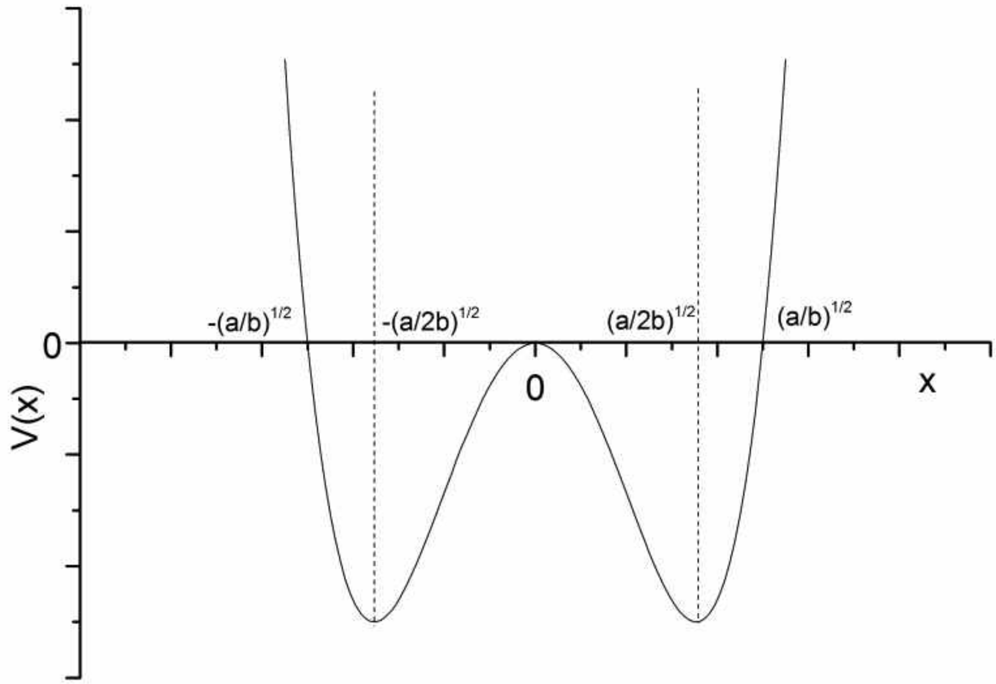
has three roots

$$x_\alpha = 0, x_\beta = -\sqrt{a/2b}, x_\gamma = \sqrt{a/2b}$$

and

$$\frac{\partial^2 V}{\partial x^2}|_{x=0} < 0, \frac{\partial^2 V}{\partial x^2}|_{x=\sqrt{a/2b}} > 0, \frac{\partial^2 V}{\partial x^2}|_{x=-\sqrt{a/2b}} > 0$$

So there is one peak at $x = 0$, and two valley at $x = \sqrt{a/2b}$ and $x = -\sqrt{a/2b}$



1. Initially, the particle has the total energy $E > 0$. Because of the damping, the total energy of the particle will gradually decrease to nearly zero after many periods. From the analysis above of the features of the potential, we know that there is a hump $V(x) = 0$ at $x = 0$. So when the particle with total energy nearly zero comes towards $x = 0$, the velocity will decelerate to nearly zero. And also due to the zero acceleration at $x = 0$, it takes very long time for the particle to pass the point $x = 0$.

$$E \to 0^+ \Rightarrow T \to \infty \qquad (1)$$

2. When $i \to i_0$, from the analysis above we know that the $T_i \to \infty$ and the total energy of the particle $E \to 0$. The particle oscillates between $x = -\sqrt{a/b} - \Delta$ and $x = \sqrt{a/b} + \Delta$, where $\Delta \to 0^+$. The particle spends nearly all the time passing the area near $x = 0$, compared to this period of time, the time which the particle spends on passing other areas can be neglected.So,

$$T_i \approx 2\int_{-\delta x}^{\delta x} \frac{dx}{v(x)} \qquad (2)$$

Where $\delta x > 0$ and very small.Because the damping is very light, the total energy can be treated as unchanged in one period. So

$$v(x)=\sqrt{\frac{2}{m}(E_i+ax^2-bx^4)} \qquad (3)$$

and hence,

$$T_i\approx 2\sqrt{\frac{m}{2}}\int_{-\delta x}^{\delta x}\frac{dx}{\sqrt{E_i+ax^2-bx^4}} \qquad (4)$$

For $(\delta x)^2 \ll \frac{a}{b}$, bx^4 can be neglected.

$$T_i\approx 2\sqrt{\frac{m}{2}}\int_{-\delta x}^{\delta x}\frac{dx}{\sqrt{E_i+ax^2}} \qquad (5)$$

$$\int_{-\delta x}^{\delta x}\frac{dx}{\sqrt{E_i+ax^2}}=\ln(\delta x+\sqrt{(\delta x)^2+E_i/a})-\ln(-\delta x+\sqrt{(\delta x)^2+E_i/a}) \qquad (6)$$

Because when $i\rightarrow i_0$, $E_i\rightarrow 0^+$. Hence,

$$\ln\left(\delta x+\sqrt{(\delta x)^2+E_i/a}\right)\approx\ln(2\delta x) \qquad (7)$$

,

$$\ln\left(-\delta x+\sqrt{(\delta x)^2+E_i/a}\right)\approx\ln(\frac{E_i}{2a\delta x}) \qquad (8)$$

Hence,

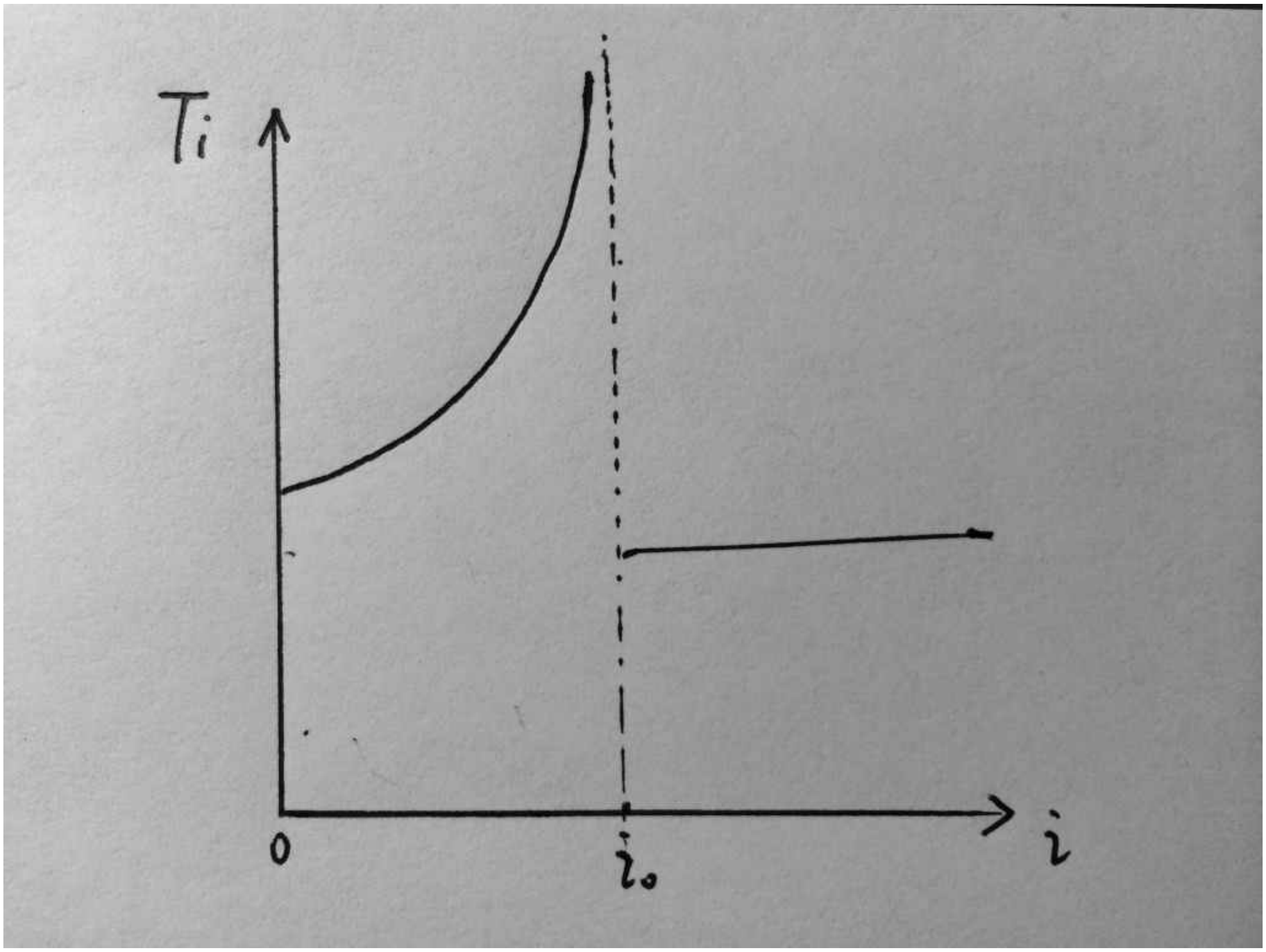
$$T_i\sim-\ln(E_i) \qquad (9)$$

Because the damping is very light, we can reasonably consider the E decreases linearly with i with the rate β, $E_i=E_{initial}-\beta i$. Also $E_{initial}=\beta i_0$, hence, $E_i=\beta(i_0-i)$. And

\[T_{i}\sim -\ln (i-i_0),~~(i

$$T_i\sim-\ln(i-i_0),\ \ (i<i_0) \qquad (10)$$

3. When $i<i_0$, the periods gradually increase with the i and near $i=i_0$, $T_i\sim -ln(i-i_0)$. When $i=i_0$, $T_i\rightarrow\infty$. When $i>i_0$, $E_i<0$, hence the particle can never pass point $x=0$ and will oscillate in the valley around the point either $x=-\sqrt{a/2b}$ or $x=\sqrt{a/2b}$. Hence, the sketch of T_i as a function of i is



3 thoughts on “M01M.2”

October 8, 2013 at 1:31 am

Very good.
Several remarks:

1) In order for the solution to be perfect, not just good, it would be great to do an estimate of your δx from

the equation (2). Namely, a) to show for which values of δx the particle indeed stays most of the time in the region $-\delta x < x < \delta x$, b) to find for which values of δx you indeed can neglect bx^4 in passing from (4) to (5), c) to check that there exists a region of δx satisfying both (a) and (b) -- that would be the region of values appropriate for your solution.

In most cases the solution as you have written it would be physically reasonable. But in my opinion it's nice to do an estimate for completeness.

October 8, 2013 at 1:36 am

2) Remark about the paragraph between (9) and (10). Even though the damping is very light, the damping constant β can change after a very long time and can be different for high energies and for energies $E \approx 0$ (in other words, it can slightly depend on i). Thus, strictly speaking, the liner approximation $E = \beta(i_0 - i)$ is valid only for small region of energies near zero (with a fixed β at this region), and it's not precisely correct to write $E = E_{initial} - \beta i$ for the whole range of i. However $E \approx 0$ is precisely the region you're interested in in this problem, so it doesn't affect the solution.

3) You sketch in part (3) is not quite right for $i > i_0$. Think more about it: the period doesn't immediately become small after passing i_0. What is the value of the limit $\lim_{i-i_0 \to +0} T(i)$?
