M01M.2 - Particle in Anharmonic Potential

Solution to M01M.2 — Particle in Anharmonic Potential

For small damping, we can assume that the total energy during one oscillation is approximately constant and that the energy decreases a fixed amount from one complete oscillation to the next. Letting x_- and x_+ be the turning points of the motion, we note that

$$T = \int_{x_-}^{x_+} \frac{\mathrm{d}x}{v(x)} = \sqrt{2m} \int_0^{x_+} \frac{\mathrm{d}x}{\sqrt{E + ax^2 - bx^4}}$$

For very small energy, the velocity will go to zero near $x = 0$. However, unlike a harmonic potential, the acceleration also vanishes at this point, resulting in the particle spending a very long time near the origin. In particular, if $E = 0$, then $T \sim \int_0^{x_+} \frac{\mathrm{d}x}{x\sqrt{a - bx^2}}$ which diverges by a well known convergence test from real analysis. This explains why the period becomes very large for some number of oscillations when the total energy is very close to zero.

Now, write

$$T \sim \int_0^{\epsilon} \frac{\mathrm{d}x}{v(x)} + \int_{\epsilon}^{x_+ - \delta} \frac{\mathrm{d}x}{v(x)} + \int_{x_+ - \delta}^{x_+} \frac{\mathrm{d}x}{v(x)}$$

The second integral can be bounded by $\max(\frac{1}{v(\epsilon)}, \frac{1}{v(x_+ - \delta)})(x_+ - (\delta + \epsilon))$ where this product is finite in the limit of $E \to 0$, so this term will always converge. Moreover, the third term will not diverge since acceleration does not vanish even as the velocity goes to zero. Hence, we are justified in claiming that there exists $\epsilon \ll \sqrt{\frac{a}{b}}$ such that (dropping coefficients since we are only concerned with scaling):

$$T \approx \int_0^{\epsilon} \frac{\mathrm{d}x}{\sqrt{E + ax^2}}$$

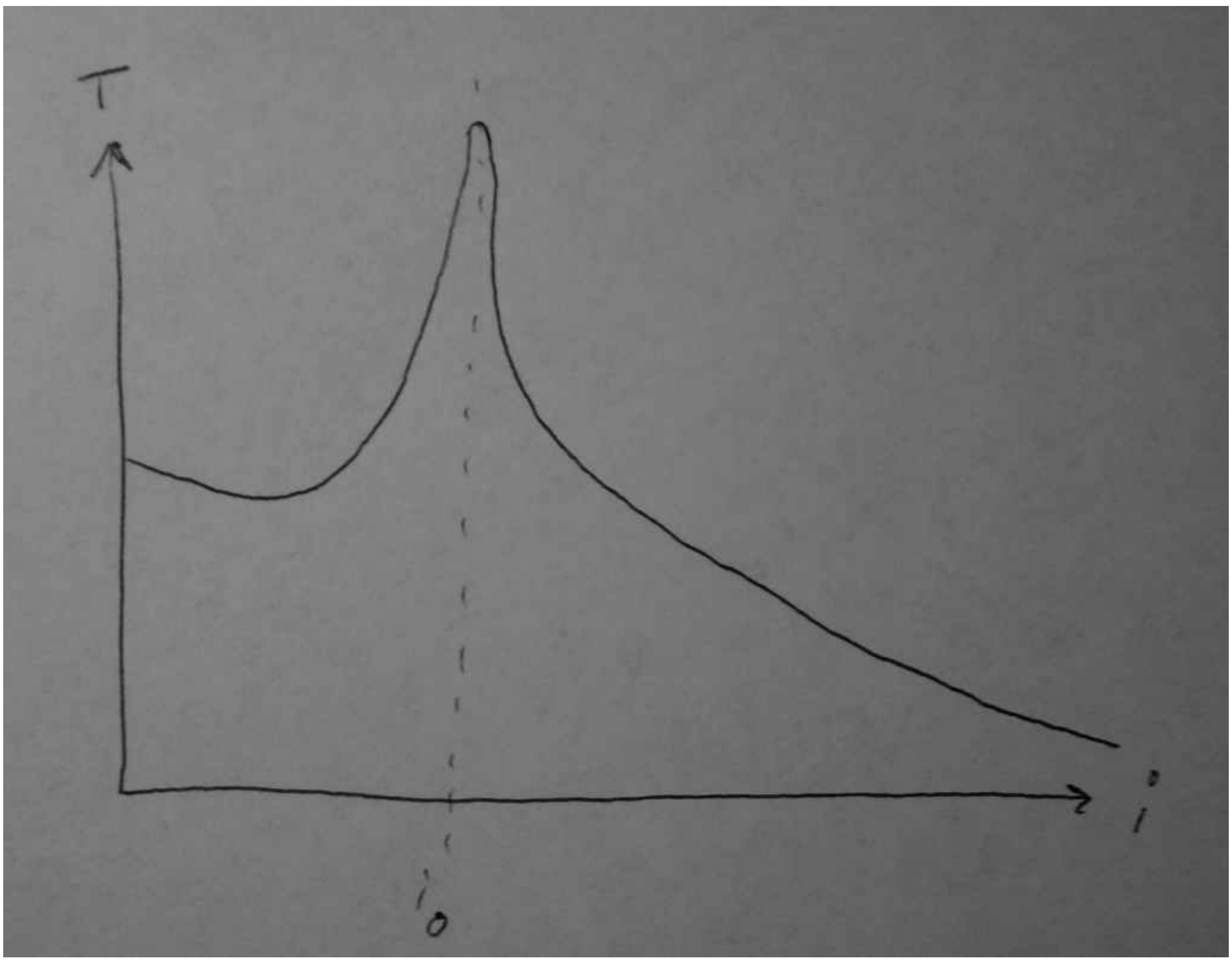
Letting $\alpha^2 = \frac{E}{a}$, this integral can be evaluated exactly as

$$T \sim \log\left(\epsilon + \sqrt{\epsilon^2 + \alpha^2}\right)\Big|_0^{\epsilon} \approx \log\frac{2\epsilon}{\alpha} \sim -\log E$$

For very light damping, we can expect that a fixed amount of energy will be lost during each period, so for energies close to zero, $E_i = E_{\text{small}} - \Delta E$, where E_{small} is some energy near zero where this linear damping approximation becomes valid. Hence, $E_i = \Delta E i_0 - i\Delta E$ so that $|E_i| \sim |i - i_0|$. Substituting this into our expression above gives us the scaling law which diverges as $i \to i_0$:

$$T \sim -\log|i - i_0|$$

A rough sketch of how the period changes with each successive oscillation is shown below.



One thought on “M01M.2 - Particle in Anharmonic Potential”

December 3, 2013 at 4:58 am

OK, that is correct.