M01M.1

Solution to M01M.1 — Massive Spring

a) Consider a segment between $[x, x+dx]$, forces applied to this segment are tension force from the upper part $T(x)$ and the lower part $-T(x+dx)$ and the gravity $-\rho g dx$. For the segment to be in equilibrium:

$$T(x) - T(x+dx) - \rho g dx = 0 \tag{1}$$

Therefore,

$$\frac{dT}{dx} = -\rho g \tag{2}$$

On the other hand, we can write the tension force starting from the properties of spring:

$$T(x) = k(x)(s(x+dx) - s(x)) \tag{3}$$

where $k(x) * dx/L = k_0$, the reason being that the whole spring can be considered as spring of length dx in series. Thus,

$$T(x) = k_0 L \frac{ds}{dx} \tag{4}$$

Therefore,

$$\frac{d^2 s}{dx^2} = -\frac{\rho g}{k_0 L} \tag{5}$$

By applying boundary conditions,

$$T(L) = 0 \tag{6}$$
$$s(0) = 0 \tag{7}$$

Finally we get,

$$s(x) = \frac{\rho g}{k_0}(-\frac{x^2}{2L} + x) \tag{8}$$

b) If we suddenly "turn off" gravity, we will get,

$$m\frac{\partial^2 s}{\partial t^2} = \frac{\partial T}{\partial x} \tag{9}$$

Therefore,

$$\frac{\partial^2 s}{\partial t^2} = \frac{k_0 L}{\rho}\frac{\partial^2 s}{\partial x^2} \tag{10}$$

Here, we can recognize the wave equation whose solution is the combination of an down-travelling function $f(x + \sqrt{\frac{k_0 L}{\rho}}t)$ and a up-travelling $g(x - \sqrt{\frac{k_0 L}{\rho}}t)$.

By applying boundary conditions,

$$s(x,0) = \frac{\rho g}{k_0}(-\frac{x^2}{2L} + x) \tag{11}$$
$$\frac{\partial s}{\partial t}(x,0) = 0 \tag{12}$$

Therefore,

$$f(x) = -\frac{\rho g}{2k_0}(\frac{x^2}{2L} - x) + \frac{A}{2} \tag{13}$$

$$g(x) = -\frac{\rho g}{2k_0}(\frac{x^2}{2L} - x) - \frac{A}{2} \tag{14}$$

Finally, with $s(x,t) = f(x + \sqrt{\frac{k_0 L}{\rho}} t) + g(x - \sqrt{\frac{k_0 L}{\rho}} t)$, we get

$$s(x,t) = -\frac{\rho g}{4k_0 L}(x + \sqrt{\frac{k_0 L}{\rho}} t)^2 - \frac{\rho g}{4k_0 L}(x - \sqrt{\frac{k_0 L}{\rho}} t)^2 + -\frac{\rho g}{k_0} x \tag{15}$$

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One thought on "M01M.1"

December 8, 2013 at 8:38 pm

Part (a) is correct.
In (b) you applied initial conditions, not boundary conditions.
Note that boundary conditions $s(0,t) = 0$ and $s_x(L,t) = 0$ are not satisfied by your solution. Think why and fix it.
If needed, I can provide a hint.