M01M.1

Solution to M01M.1— Spherical Half-Filled Capacitor

a) According to balance of a short part of the spring, $T(x+dx) - T(x) = -\rho g$, thus $\frac{dT}{dx} = -\rho g$, combined with boundary condition tension $T(x=L) = 0$, we have

$$T(x) = \rho g(L-x) \tag{1}$$

Hooke's Law gives $T(x) = k_d ds(x)$, where $k_d = k\frac{L}{dx}$, then

$$T(x) = kL\frac{ds(x)}{dx} = \rho g(L-x) \tag{2}$$

Noticing that $s(x=0)=0$, solve (2)

$$s(x) = \frac{\rho g}{kL}(Lx - \frac{1}{2}x^2) \tag{3}$$

b) Apply Newton's 2nd Law to a short part of the spring

$$\frac{dT}{dx} = \rho\frac{\partial^2 s(x,t)}{\partial t^2} \tag{4}$$

Insert the relation between tension and displacement (2)

$$\frac{\partial^2 s(x,t)}{\partial t^2} - v^2\frac{\partial^2 s(x,t)}{\partial x^2} = 0 \qquad \text{where } v^2 = \frac{kL}{\rho} \tag{5}$$

Boundary condition gives

$$s(x,t)|_{x=0} = 0 \tag{6}$$
$$\frac{\partial s(x,t)}{\partial t}|_{x=L} = 0 \tag{7}$$

Initial condition gives

$$s(x,0) = \frac{\rho g}{kL}(Lx - \frac{1}{2}x^2) \tag{8}$$
$$\frac{\partial s}{\partial x}|_{t=0} = 0 \tag{9}$$

Solve equation (5) using separation of variables, general solution is

$$s(x,t) = Acos(\omega t + \alpha)cos(\kappa x + \beta) \qquad \text{where } \omega v = \kappa \tag{10}$$

use boundary condition, we get

$$\kappa_n = (n - \frac{1}{2})\pi/L \tag{11}$$
$$\beta_n = \pi/2 \qquad n = 1,2,3... \tag{12}$$
$$s(x,t) = \sum_{n=1}^{+\infty} A_n sin(\kappa_n x)cos(\omega_n t + \alpha_n) \tag{13}$$

Apply initial conditions

$$\alpha_n = 0 \tag{14}$$
$$A_n = \frac{2}{L}\int_0^L sin(\kappa_n x)(\frac{\rho g}{2kL})(Lx - \frac{1}{2}x^2)dx \tag{15}$$
$$= \frac{\rho g}{kL^2\kappa_n^3} \tag{16}$$

Thus, the subsequent motion $s(x,t)$ of the spring is

$$s(x,t) = \sum_{n=1}^{+\infty} \frac{\rho g}{L^2\kappa_n^3} sin(\kappa_n x)cos(\omega_n t) \tag{17}$$

3 thoughts on “M01M.1”

M
October 3, 2013 at 6:12 pm

Also, you can find a closed expression for $s(x,t)$ in terms of elementary functions rather than just a general sum over harmonics.

M
October 2, 2013 at 4:54 pm

Good.
Just one typo: in (7) there should be an x derivative.
Also, the minus sign in (16) seems suspicious -- looks like integration gives plus.

Q
October 2, 2013 at 9:29 pm

the minus sign in (16) should be positive in fact.
