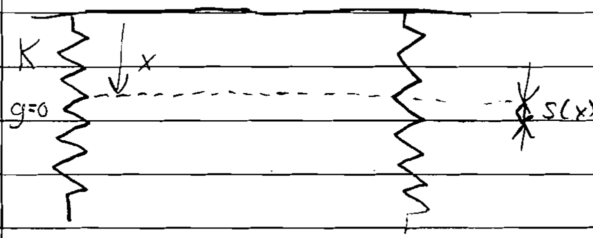
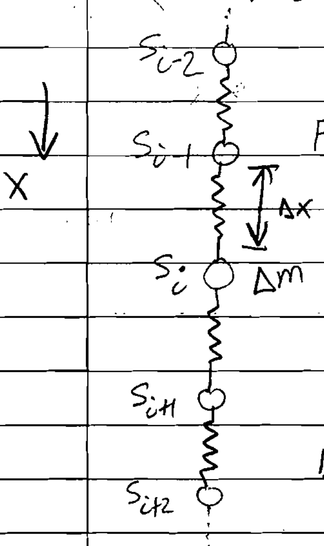
May 2001 #1



$g \downarrow$ unstretched length L, mass per unit length ρ

$\Rightarrow M = \rho L$

Model spring as many elements Δm connected by massless springs then let $N \to \infty$



Force on mass i: $F = \Delta m \cdot g + k(s_{i+1} - s_i) - k(s_i - s_{i-1})$

$$\Delta m \ddot{s}_i = \Delta m g + k(s_{i+1} - 2s_i + s_{i-1})$$

transition to continuum limit:

$$s_{i+1} - 2s_i + s_{i-1} \to (\Delta x)^2 \frac{d^2 s}{dx^2}$$

$\Delta x = \dfrac{L}{N}$ $\quad k = NK \quad$ K is total spring constant (N springs in series)

$\Delta m = \dfrac{M}{N} = \dfrac{\rho L}{N}$ (assuming the spring doesn't stretch much, so Δx doesn't change much)

a) steady state: $\ddot{s}_i = 0 \quad \Delta m g + k(\Delta x)^2 \dfrac{d^2 s}{dx^2} = 0$

$$\frac{d^2 s}{dx^2} = \frac{-\Delta m g}{k(\Delta x)^2} = \frac{-\rho L g}{N} \frac{1}{KN} \cdot \frac{N^2}{L^2} = \frac{-\rho g}{LK}$$

$s(0) = 0$: the spring part connected to wall doesn't move

Force on bottom element: $\Delta m g - k(s_N - s_{N-1}) = 0$

$s_N - s_{N-1} \to \Delta x \, s'(L)$

$$\frac{\Delta m g}{k \Delta x} = s'(L) = \frac{\rho L g}{N} \frac{1}{KN} \frac{N}{L} = \frac{\rho g}{KN} \to 0 \quad \text{as } N \to \infty$$

$s'(L) = 0$

$s' = \dfrac{-\rho g}{LK} x + C_1 \qquad s'(L) = 0 = \dfrac{-\rho g L}{LK} + C_1 \Rightarrow C_1 = \dfrac{\rho g}{K}$

$s'(x) = \frac{\rho g}{K}\left(1 - \frac{x}{L}\right)$

$s = \frac{\rho g}{K}\left(x - \frac{x^2}{2L}\right)$ constant $=0$

$= s(x,0)$

b) $g \to 0$ $\frac{\rho}{LK}\ddot{s} = s''$

expand $s(x,t)$ in eigenfunctions

initial position $\neq 0$,

initial velocity $=0$ different K than before

$\Rightarrow \cos(\omega t)\sin(kx)$

$-k^2 = -\omega^2 \frac{\rho}{LK}$ $\omega = k\sqrt{\frac{L}{\rho}K}$

$s'(L)=0 \Rightarrow \cos(kL)=0$

$kL = (n+\frac{1}{2})\pi$ $k_n = \frac{(n+\frac{1}{2})\pi}{L}$ $n=0,1,2,\ldots$

$$s(x,t) = \sum_{n=0}^{\infty} A_n \cos(\omega_n t)\sin(k_n x)$$

$$s(x,0) = \sum_{n=0}^{\infty} A_n \sin(k_n x)$$

Fourier Series: $A_n = \frac{2}{L}\int_0^L s(x,0)\sin(k_n x)\,dx$