M01M.1

(a) To find the equation of motion of the spring we first describe the force along the spring at the position which began at x by a force $F(x)$. Writing the equation of motion for a small portion of the spring, $\rho dx \ddot{s}(x) = F(x + dx) - F(x) + \rho g x$ Dividing and taking the limit as $dx \to 0$ we get. $\rho \ddot{s}(x) = \partial_x F(x) + \rho g$.

To find $F(x)$ we consider our system as divided into three parts. One is the upper part of the spring, one is the lower part of the spring, and the other is a portion of spring of length dx, which is stretched to length $S(x + dx) - S(x)$. The spring constant of such a spring can be found by noting that springs add reciprocally. Therefore if L/dx identical springs are to create a total spring of constant k they must each have constant $\kappa = kL/dx$. So, $F(x) = \kappa(s(x + dx) - s(x)) = kL\partial_x s(x)$in the limit as $dx \to 0$.

Our equation of motion becomes $\rho \ddot{s}(x,t) = kL\partial_x^2 s(x) + \rho g$.

(a) For a stationary spring, the left side vanishes, and integrating twice yields $s(x) = -\frac{\rho g}{2kL} x^2 + c_1 x + c_2$. We require $s(0) = 0$, and $F(L) = 0$, since the spring is fixed, and the end is not moving. The first condition implies that $c_2 = 0$. The second implies that $\partial_x s|_L = -\frac{\rho g}{k} + c_1$ which implies $c_1 = -\frac{\rho g}{k}$. This gives us $s(x) = \frac{\rho g}{2kL} x(2L - x)$.

(b) We now want to solve the wave equation $\partial_x^2 s - \frac{\rho}{kL} \ddot{s} = 0$ given the initial condition found in (a), and initial velocity 0. In general such an equation has solution $f_1(x + ct) + f_2(x - ct)$ where $c = \sqrt{kL/\rho}$.

The requirement that the initial velocity is zero enforces $f_1'(u) = f_2'(u), \forall u \in [0, L]$

The given $S(x, 0)$ requires $f_1(x) + f_2(x) = \frac{\rho g}{2kL} x(2L - x), \forall x \in [0, L]$.

The requirement that the initial part of the spring is fixed yields $f_1(u) + f_2(-u) = 0, \forall u \in \mathbb{R}$.

The first and second constraints yield that $f_1(x) = f_2(x) = \frac{\rho g}{4kL} x(2L - x), \forall x \in [0, L]$.

To apply the third constraint, we need consider all of $\mathbb{R}$ not merely the interval $[0, L]$. Notice that we can define $f_1(x) = f_2(x) = \frac{\rho g}{4kl} x(2L + x), \forall x \in [-L, 0]$ to enforce this condition on $[-L, L]$.

How do we extend our function to the rest of the real line? One obvious way is to make the function we have here periodic. That is, to define

$$f_1(x) = f_2(x) = \begin{cases} \dfrac{\rho g}{4kL}(x - 4n)(4L(2n+1) - x), \forall x \in [4nL, 2(2n+1)L] \\ \dfrac{\rho g}{4kl}(x - 2(2n+1))(8nL + x), \forall x \in [2(2n+1)L, 4(n+1)L] \end{cases}$$

for $n \in \mathbb{Z}$. Note furthermore that this function is twice differentiable even at the points where we have stitched our two functions together. However, here the second derivative is discontinuous. i.e. $f_1 \notin C^2(\mathbb{R})$.

This is basically what we do when solving a wave on a string by Fourier series (e.g.). We suppose our solution is extended off of the given domain by making it periodic. The difference is that for sinusoidal waves the construction is automatic.

3 thoughts on "M01M.1"

P

October 16, 2013 at 11:45 am

It seems alarming that this new equation no longer has the property that $s(0,t)=0$.

M

December 4, 2013 at 3:14 am

Indeed, your solution doesn't satisfy boundary conditions.
The reason is that initial conditions as you've used them define $f_1(x)$ and $f_2(x)$ only for $0<x<L$.
However, for the solution $s(x,t)=f_1(x-vt)+f_2(x+vt)$ to be valid for all t, you need to know functions f_1 and f_2 outside of the interval [0,L]. You need to define them there in a smart way such that the boundary condition will be satisfied.

M

October 8, 2013 at 3:09 pm

Please redo this problem.
You cannot assume that the upper part of the spring behaves as a massless spring with some effective spring constant when grvity is present. The reason is that there is some non-uniform extension of the spring due to gravity. For example, your equations would imply that S(L)=0, which is obviously not true.
